INNMO 2023/2

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Twitch Solves ISL

Episode 112

Problem

Suppose $a_0, \dots, a_{100}$ are positive reals. Consider the following polynomial for each k in $\{0, 1, \dots, 100\}$:

$$a_{100+k}x^{100} + 100a_{99+k}x^{99} + a_{98+k}x^{98} + a_{97+k}x^{97} + \cdots + a_{2+k}x^2 + a_{1+k}x + a_k,$$

where indices are taken modulo 101. Show that it is impossible that each of these 101 polynomials has all its roots real.

Video

https://youtu.be/9WdIHzvXF64

External Link

https://aops.com/community/p26888629

Solution

Assume for contradiction.

Take the reciprocal polynomial

$$Q_k(x) = a_k X^{100} + a_{k+1} X^{99} + a_{k+2} X^{98} + \dots$$

which has also 101 real roots for each k. Differentiate it 98 times:

$$\begin{aligned} Q_k^{(98)}(x) &= \frac{100!}{2} a_k X^2 + 99! a_{k+1} X + 98! a_{k+2} \\ &= 98!(4950 a_k X^2 + 99 a_{k+1} X + a_{k+2}). \end{aligned}$$

By Rolle's theorem this should still have 2 real roots. Ergo, the discriminant is nonnegative:

$$(99a_{k+1})^2 \ge 4 \cdot 4950 a_k a_{k+2} \implies a_{k+1}^2 > a_k a_{k+2}$$

since $4 \cdot 4950 > 99^2$. But multiplying all of these together gives a contradiction