# INMO 2023/2 

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## Twitch Solves ISL

Episode 112

## Problem

Suppose $a_{0}, \ldots, a_{100}$ are positive reals. Consider the following polynomial for each $k$ in $\{0,1, \ldots, 100\}$ :

$$
a_{100+k} x^{100}+100 a_{99+k} x^{99}+a_{98+k} x^{98}+a_{97+k} x^{97}+\cdots+a_{2+k} x^{2}+a_{1+k} x+a_{k}
$$

where indices are taken modulo 101. Show that it is impossible that each of these 101 polynomials has all its roots real.

## Video

https://youtu.be/9WdIHzvXF64

## External Link

## Solution

Assume for contradiction.
Take the reciprocal polynomial

$$
Q_{k}(x)=a_{k} X^{100}+a_{k+1} X^{99}+a_{k+2} X^{98}+\ldots
$$

which has also 101 real roots for each $k$. Differentiate it 98 times:

$$
\begin{aligned}
Q_{k}^{(98)}(x) & =\frac{100!}{2} a_{k} X^{2}+99!a_{k+1} X+98!a_{k+2} \\
& =98!\left(4950 a_{k} X^{2}+99 a_{k+1} X+a_{k+2}\right)
\end{aligned}
$$

By Rolle's theorem this should still have 2 real roots. Ergo, the discriminant is nonnegative:

$$
\left(99 a_{k+1}\right)^{2} \geq 4 \cdot 4950 a_{k} a_{k+2} \Longrightarrow a_{k+1}^{2}>a_{k} a_{k+2}
$$

since $4 \cdot 4950>99^{2}$. But multiplying all of these together gives a contradiction

