ELMO Revenge 2022/1

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Twitch Solves ISL

Episode 112

Problem

Let ABC and DBC be triangles with incircles touching at a point P on BC. Points A, D lie on the same side of BC and $DB < AB < DC < AC$. The bisector of $\angle BDC$ meets line AP at X, and the altitude from A meets DP at Y. Point Z lies on line XY so $ZP \perp BC$. Show the reflection of A over BC is on line ZD.

Video

https://youtu.be/QVD4kFNyCHQ

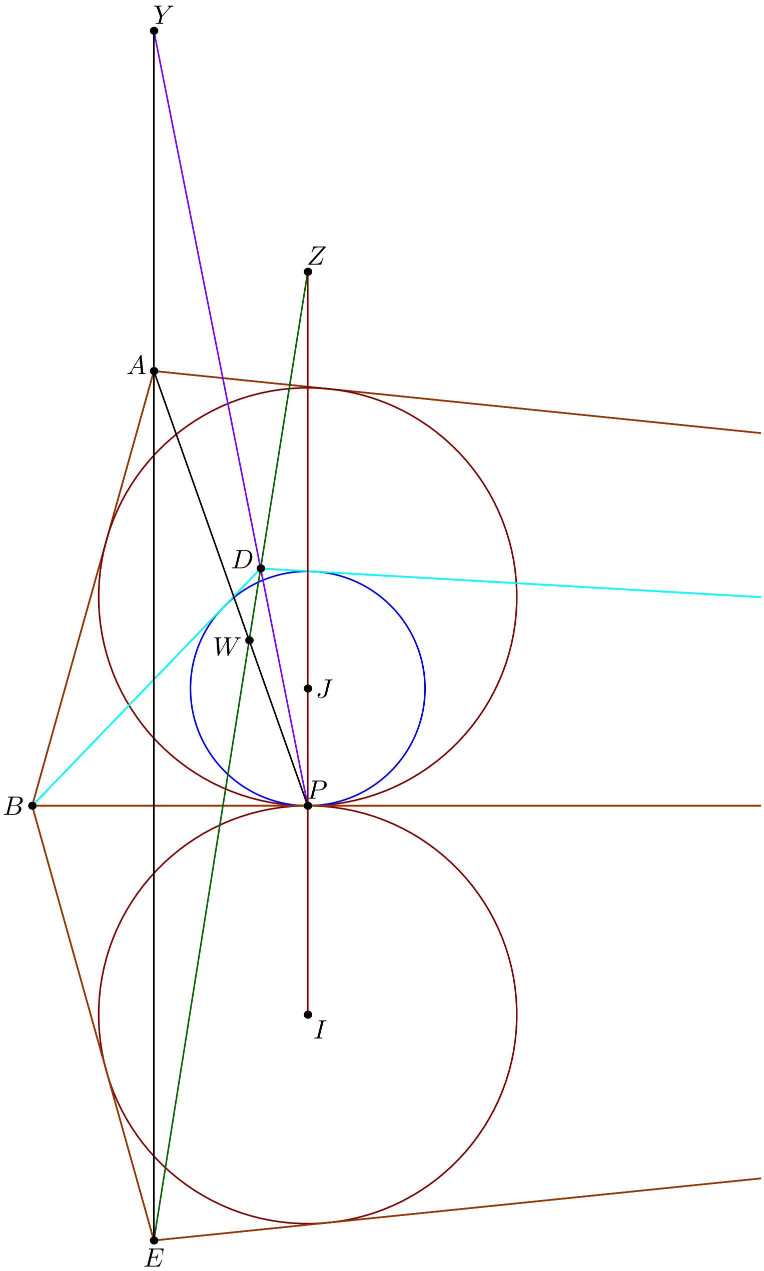
External Link

https://aops.com/community/c6h2881590p25619349

Solution

Intended solution Let A' be the reflection of A over BC. Then the points A, D, P, A' lie on a hyperbola with foci at B and C by construction. Then apply Pascal to $DDPPAA'$.

Synthetic solution Let E be the reflection of A across $\overline{BC}$. Let I and J denote the incenters of $\triangle EBC$ and $\triangle DBC$. Redefine $Z = \overline{DE} \cap \overline{IPJ}$; then the problem is to show X, Y, Z are collinear.



To interpret the condition about P:

Claim. $(IJ; PZ) = -1$.

Proof. Because the condition on P forces $BD + CE = BE + CD$, there is an incircle for $BDCE$. Hence by Monge theorem, Z coincides with the exsimilicenter of (I) and (J). As I is the insimilicenter, the conclusion follows. $\square$

Next, let $W = \overline{ZDE} \cap \overline{AP}$.

Claim. Triangles DJP and AWE are perspective, i.e. the lines $\overline{DA}$, $\overline{JW}$, $\overline{PE}$ are concurrent.

Proof. We have $-1 = (ZP; JI) \stackrel{W}{=} (E, A; \overline{JW} \cap \overline{AE}, Y)$. Then it follows by looking at complete quadrilateral $ADPEWY$. $\square$

Now by Desargue's theorem, it follows $\overline{DJ} \cap \overline{AW} = X$, $\overline{DP} \cap \overline{AE} = Y$, $\overline{JP} \cap \overline{WE} = Z$ are collinear.