USAMTS 4/3/34

Evan Chen

Twitch Solves ISL

Episode 111

Problem

Let ω be a circle with center O and radius 10, and let H be a point such that $OH = 6$. A point P is called snug if, for all triangles ABC with circumcircle ω and orthocenter H, we have that P lies on $\triangle ABC$ or in the interior of $\triangle ABC$. Find the area of the region consisting of all snug points.

Video

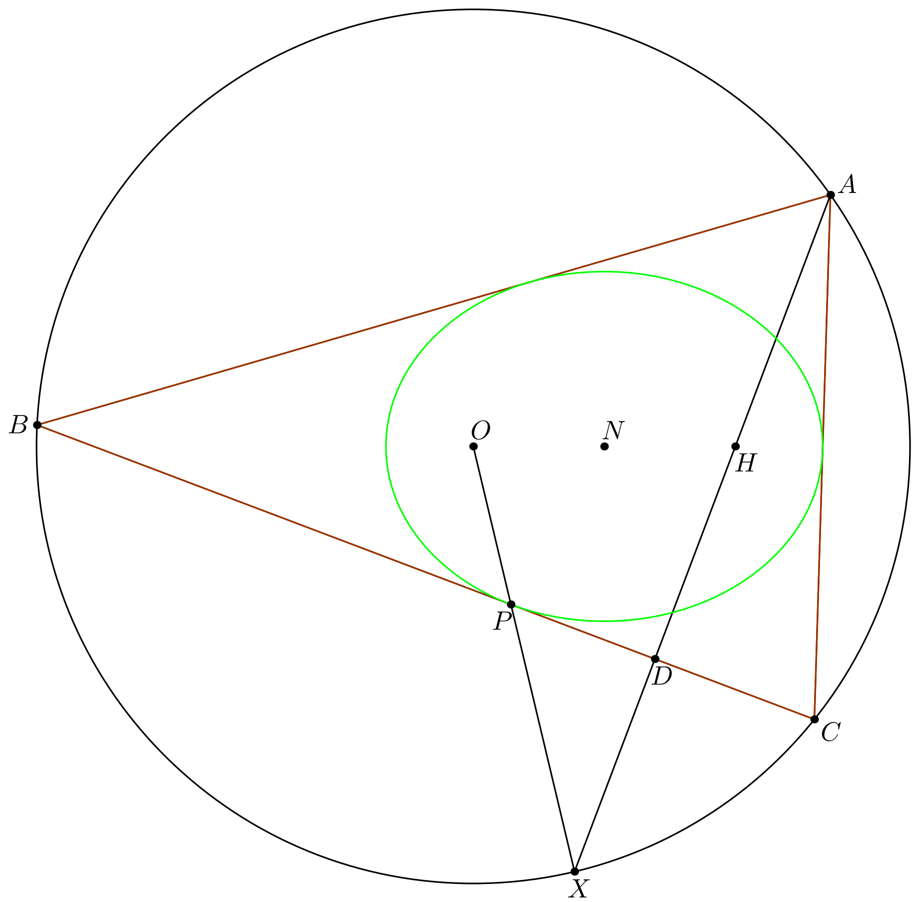
https://youtu.be/V58ZweJKKGA

External Link

https://aops.com/community/p26832931

Solution

We claim that the locus is the ellipse $\mathcal{E}$ whose foci are at O and H and which contains every point S satisfying $OS + SH \le 10$ (i.e. has major axis of length 10).



Claim. $\mathcal{E}$ is always an in-ellipse of $\triangle ABC$, and hence every point on or inside $\mathcal{E}$ is snug.

Proof. Since H is inside ω, it follows ABC is acute. Now, let X be the reflection of H over BC, and let $P = \overline{BC} \cap \overline{OX}$. Then

$$OP + PH = OP + PX = OX = 10$$

so P lies on $\mathcal{E}$. Moreover, because of the "river reflection" problem (or just noting that $\measuredangle OPB = \measuredangle CPH$), it follows that line BC is actually *tangent* to $\mathcal{E}$ at P.

Similarly, $\mathcal{E}$ is tangent to lines CA and AB. So $\mathcal{E}$ is an in-ellipse of $\triangle ABC$. $\square$

Conversely, suppose S is a point not inside the ellipse. Then line segment NS intersects the ellipse at some point P. For continuity reasons, one can choose ABC such that P is the tangency point of BC. Then line BC separates N and P, and N is obviously inside ABC, so P is outside ABC and hence not snug.