# USAMTS 4/2/34

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Twitch Solves ISL

Episode 111

## Problem

Fix an integer $k \geq 2$. Find the smallest positive integer $c_k$ such that a toroidal $k \times k$ board can be colored with one of $c_k$ colors, where orthogonal and diagonal neighbors of a point are different colors.

## Video

https://youtu.be/uUJ9BlrjJHg

## External Link

https://aops.com/community/p26618826

## Solution

The answer is

$$c_k = \begin{cases} 9 & \text{if } k = 3 \\ 4 & \text{if } k \text{ is even} \\ 5 & \text{if } k \geq 5 \text{ is odd.} \end{cases}$$

When $k = 3$, nine colors are both necessary and sufficient because the graph we are trying to color is $K_9$ (every two cells are the same color).

When $k$ is even, a coloring using $(x \bmod 2, y \bmod 2)$ works. It's also best possible because every cell of each $2 \times 2$ must be a different color.

The main interesting is $k \geq 5$ odd.

**Claim.** At least five colors are necessary when $k$ is odd.

*Proof.* In fact one cannot even color a $2 \times k$ toroidal grid with four colors, since if the first column is say red/pink and the second column is blue/purple then the columns will alternate red/pink and blue/purple. □

On the other hand, a construction for $k = 13$ that generalizes easily is shown below.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| R | G | Y | K | B | R | G | R | G | R | G | R | G |
| Y | K | B | R | G | Y | K | Y | K | Y | K | Y | K |
| B | R | G | Y | K | B | R | B | R | B | R | B | R |
| G | Y | K | B | R | G | Y | G | Y | G | Y | G | Y |
| K | B | R | G | Y | K | B | K | B | K | B | K | B |
| R | G | Y | K | B | R | G | R | G | R | G | R | G |
| Y | K | B | R | G | Y | B | Y | B | Y | B | Y | B |
| R | G | Y | K | B | R | G | R | G | R | G | R | G |
| Y | K | B | R | G | Y | B | Y | B | Y | B | Y | B |
| R | G | Y | K | B | R | G | R | G | R | G | R | G |
| Y | K | B | R | G | Y | B | Y | B | Y | B | Y | B |
| R | G | Y | K | B | R | G | R | G | R | G | R | G |
| Y | K | B | R | G | Y | B | Y | B | Y | B | Y | B |