

# USAMTS 4/2/34

Evan Chen

TWITCH SOLVES ISL

Episode 111

## Problem

Fix an integer  $k \geq 2$ . Find the smallest positive integer  $c_k$  such that a toroidal  $k \times k$  board can be colored with one of  $c_k$  colors, where orthogonal and diagonal neighbors of a point are different colors.

## Video

<https://youtu.be/uUJ9BlrjJHg>

## External Link

<https://aops.com/community/p26618826>

## Solution

The answer is

$$c_k = \begin{cases} 9 & \text{if } k = 3 \\ 4 & \text{if } k \text{ is even} \\ 5 & \text{if } k \geq 5 \text{ is odd.} \end{cases}$$

When  $k = 3$ , nine colors are both necessary and sufficient because the graph we are trying to color is  $K_9$  (every two cells are the same color).

When  $k$  is even, a coloring using  $(x \bmod 2, y \bmod 2)$  works. It's also best possible because every cell of each  $2 \times 2$  must be a different color.

The main interesting is  $k \geq 5$  odd.

**Claim.** At least five colors are necessary when  $k$  is odd.

*Proof.* In fact one cannot even color a  $2 \times k$  toroidal grid with four colors, since if the first column is say red/pink and the second column is blue/purple then the columns will alternate red/pink and blue/purple.  $\square$

On the other hand, a construction for  $k = 13$  that generalizes easily is shown below.

R	G	Y	K	B	R	G	R	G	R	G	R	G
Y	K	B	R	G	Y	K	Y	K	Y	K	Y	K
B	R	G	Y	K	B	R	B	R	B	R	B	R
G	Y	K	B	R	G	Y	G	Y	G	Y	G	Y
K	B	R	G	Y	K	B	K	B	K	B	K	B
R	G	Y	K	B	R	G	R	G	R	G	R	G
Y	K	B	R	G	Y	B	Y	B	Y	B	Y	B
R	G	Y	K	B	R	G	R	G	R	G	R	G
Y	K	B	R	G	Y	B	Y	B	Y	B	Y	B
R	G	Y	K	B	R	G	R	G	R	G	R	G
Y	K	B	R	G	Y	B	Y	B	Y	B	Y	B
R	G	Y	K	B	R	G	R	G	R	G	R	G
Y	K	B	R	G	Y	B	Y	B	Y	B	Y	B