# ELMO Revenge 2022/2 <br> Evan Chen 

Twitch Solves ISL

Episode 111

## Problem

Find all ordered pairs of integers $(x, y)$ such that

$$
x y\left(x^{2} y^{2}-12 x y-12 x-12 y+2\right)=(2 x+2 y)^{2} .
$$

## Video

https://youtu.be/V58ZweJKKGA

## External Link

https://aops.com/community/p25620454

## Solution

The answer is $(-1,-2),(-2,-1)$ and $(0,0)$. They work; we prove that's all.

First motivated solution. Let $a=x+y$ and $b=x y$. Then this becomes

$$
b\left(b^{2}-12 b-12 a+2\right)=(2 a)^{2} \Longrightarrow 4 \cdot a^{2}+12 b \cdot a-b\left(b^{2}-12 b+2\right)=0 .
$$

As a quadratic in $a$, the discriminant is

$$
(12 b)^{2}+4 \cdot 4 b\left(b^{2}-12 b+2\right)=16 b \cdot\left(b^{2}-3 b+2\right)=16 b(b-1)(b-2)
$$

but it should also be a perfect square. This only happens if $b \in\{0,1,2\}$. From here we recover the solution set by some casework.

Second comedic solution. Write it as

$$
(x y+x+y)^{4}=(x y+x)^{4}+(x y+y)^{4}
$$

and apply Fermat's Last theorem.
Third solution (generalization). We solve the more general equation $x y \mid 4\left(x^{2}+y^{2}\right)$.
Claim. If

$$
x y \mid 4\left(x^{2}+y^{2}\right)
$$

then we have either $x=2 y, x=4 y, y=2 x$, or $y=4 x$.
Proof. Let $x=d a, y=d b$, where $d=\operatorname{gcd}(a, b)$. Now $d^{2} a b \mid 4 d\left(a^{2}+b^{2}\right)$, so $a \mid 4 b^{2}$ and $b \mid 4 a^{2}$. This means $a, b \mid 4$ and implies the result.

Now plug each of this back in and manually check.

