ELMO Revenge 2022/2

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Twitch Solves ISL

Episode 111

Problem

Find all ordered pairs of integers (x, y) such that

$$xy(x^2y^2 - 12xy - 12x - 12y + 2) = (2x + 2y)^2.$$

Video

https://youtu.be/V58ZweJKKGA

External Link

https://aops.com/community/p25620454

Solution

The answer is $(-1,-2)$, $(-2,-1)$ and $(0,0)$. They work; we prove that's all.

First motivated solution Let $a = x + y$ and $b = xy$. Then this becomes

$$b(b^2 - 12b - 12a + 2) = (2a)^2 \implies 4 \cdot a^2 + 12b \cdot a - b(b^2 - 12b + 2) = 0.$$

As a quadratic in a, the discriminant is

$$(12b)^2 + 4 \cdot 4b(b^2 - 12b + 2) = 16b \cdot (b^2 - 3b + 2) = 16b(b-1)(b-2)$$

but it should also be a perfect square. This only happens if $b \in \{0, 1, 2\}$. From here we recover the solution set by some casework.

Second comedic solution Write it as

$$(xy + x + y)^4 = (xy + x)^4 + (xy + y)^4$$

and apply Fermat's Last theorem.

Third solution (generalization) We solve the more general equation $xy \mid 4(x^2 + y^2)$.

Claim. If

$$xy \mid 4(x^2 + y^2)$$

then we have either $x = 2y$, $x = 4y$, $y = 2x$, or $y = 4x$.

Proof. Let $x = da$, $y = db$, where $d = \gcd(a, b)$. Now $d^2ab \mid 4d(a^2 + b^2)$, so $a \mid 4b^2$ and $b \mid 4a^2$. This means $a, b \mid 4$ and implies the result. □

Now plug each of this back in and manually check.