Clock

Evan Chen

Twitch Solves ISL

Episode 110

Problem

There is a analog clock in front of you. How many positions of the hands are there, by which it is impossible to determine the time, if you do not know which hand is hour and which is minute?

Video

https://youtu.be/16UNuZTrNTc

Solution

We regard each time as an ordered pair (H, M) where $0 \le H < 12$ is an integer, $0 \le M < 60$ is real. The position of the hands measured in degrees from 12 clockwise is

$$(H, M) \mapsto \left\{ 30\left(H + \frac{M}{60}\right), 6M \right\} = \left\{ 30H + \frac{1}{2}M, 6M \right\}.$$

So we are looking for pairs (H, M) and (H', M') with

$$\begin{aligned} 60H + M &= 12M' \\ 60H' + M' &= 12M \end{aligned}$$

Solving,

$$\begin{aligned} M &= \frac{60H + 60 \cdot 12H'}{143} \\ M' &= \frac{60H' + 60 \cdot 12H}{143} \end{aligned}$$

For each $(H, H') \in \{0, 1, \dots, 11\}^2$ with $H \neq H'$ this gives two numbers both less than 60. Hence the number of positions of hands is just $\binom{12}{2} = 66$. (Or $12 \cdot 11 = 132$ if you consider the hands to be distinguishable when counting, even when indistinguishable to the viewer.)