# China 2023/5 

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Twitch Solves ISL

Episode 110

## Problem

Prove that there exists a constant $C>0$ such that: if $a_{1}<a_{2}<a_{3}<\cdots$ is an arithmetic sequence of positive integers for which $\operatorname{gcd}\left(a_{1}, a_{2}\right)$ is squarefree, then there exists a positive integer $m \leq\left(C a_{2}\right)^{2}$ such that $a_{m}$ is squarefree.

## Video

https://youtu.be/UsbnjfTOljs

## External Link

https://aops.com/community/p26787674

## Solution

Classify primes into three types:

- If a prime $p$ has $p^{2} \mid a_{2}-a_{1}$, or if $p \mid a_{2}-a_{1}$ but not $\operatorname{gcd}\left(a_{1}, a_{2}\right)$, then the prime is completely harmless; no term of the sequence is divisible by $p^{2}$
- If $p \nmid a_{2}-a_{1}$, then we say $p$ is mostly harmless.
- Otherwise, if $p \mid \operatorname{gcd}\left(a_{1}, a_{2}\right)$ and $\nu_{p}\left(a_{2}-a_{1}\right)=1$; we say $p$ is scary.

Let $D=\prod($ scary $p) \leq\left|a_{2}-a_{1}\right|<a_{2}$. Say a term $a_{i}$ is good if it's not divisible by the square of any scary prime.

Claim. Among any $D$ consecutive terms of $\left(a_{n}\right)_{n}$, exactly $\varphi(D)$ are good.
Claim. If $p$ is a mostly harmless prime, among any $p^{2} \cdot D$ consecutive terms, exactly $\varphi(D)$ good ones satisfy $p^{2} \mid a_{i}$.

Proof. By Chinese remainder theorem.
Let $N$ be large and consider only the terms $a_{1}, \ldots, a_{N}$ henceforth. If $p$ is a mostly harmless prime with $p<\sqrt{N}$, the number of good terms divisible by $p^{2}$ is at most $\varphi(D)\left\lceil\frac{N}{p^{2} D}\right\rceil$. On the other hand, if $\sqrt{N} \leq p \leq \sqrt{a_{N}}$, the number of terms divisible by $p^{2}$ is at most 1 , full stop. And if $p>\sqrt{a_{N}}$, then $p^{2}$ can't divide any terms. Therefore, the number of not-squarefree good terms is bounded by

$$
\begin{aligned}
& \sum_{p<\sqrt{N}}\left(\varphi(D) \cdot\left[\frac{N}{p^{2} D}\right\rceil\right)+\sum_{\sqrt{N} \leq p \leq \sqrt{a_{N}}} 1 \\
< & \frac{\varphi(D)}{D} N \sum_{p} \frac{1}{p^{2}}+\left(\varphi(D) \cdot O\left(\frac{\sqrt{N}}{\log N}\right)+O\left(\frac{\sqrt{N D}}{\log (N D)}\right)\right) \\
< & \frac{\varphi(D)}{2 D} N+\varphi(D) \cdot O\left(\frac{\sqrt{N}}{\log N}\right)
\end{aligned}
$$

where we have used three well-known facts: $\sum_{p \text { prime }} p^{-2}<\frac{1}{2}$, the prime number theorem, and the inequality $0.01 \sqrt{D} \leq \varphi(D)$ which is valid for any $D \geq 1$ (and is proved by multiplicativity of $\varphi(x) / x$, and checking at prime powers).

On the other hand, the number of good terms was at least

$$
\varphi(D) \cdot\left\lfloor\frac{N}{D}\right\rfloor>\frac{\varphi(D)}{D} N-\varphi(D)
$$

So we will have a squarefree term as long as

$$
\frac{\varphi(D)}{2 D} N>\varphi(D) \cdot O\left(\frac{\sqrt{N}}{\log N}\right)
$$

which is certainly true if $N>O(\sqrt{D})$. As $D<a_{2}$, we're done.

