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Twitch Solves ISL

Episode 110

Problem

Prove that there exist $C > 0$, which satisfies the following conclusion: if $a_1 < a_2 < a_3 < \dots$ is an arithmetic sequence of positive integers for which $\gcd(a_1, a_2)$ is squarefree, then there exists a positive integer $m \le (Ca_2)^2$ such that a_m is squarefree.

Video

https://youtu.be/UsbnjfT0ljs

External Link

https://aops.com/community/p26787674

Solution

Classify primes into three types:

- If a prime p has $p^2 \mid a_2 - a_1$, or if $p \mid a_2 - a_1$ but not $\gcd(a_1, a_2)$, then the prime is *completely harmless*; no term of the sequence is divisible by p^2
- If $p \nmid a_2 - a_1$, then we say p is **mostly harmless**.
- Otherwise, if $p \mid \gcd(a_1, a_2)$ and $\nu_p(a_2 - a_1) = 1$; we say p is **scary**.

Let $D = \prod(\text{scary } p) \le |a_2 - a_1| < a_2$. Say a term a_i is *good* if it's not divisible by the square of any scary prime.

Claim. Among any D consecutive terms of $(a_n)_n$, exactly $\varphi(D)$ are good.

Claim. If p is a mostly harmless prime, among any $p^2 \cdot D$ consecutive terms, exactly $\varphi(D)$ good ones satisfy $p^2 \mid a_i$.

Proof. By Chinese remainder theorem. □

Let N be large and consider only the terms $a_1, \dots, a_N$ henceforth. If p is a mostly harmless prime with $p < \sqrt{N}$, the number of good terms divisible by p^2 is at most $\varphi(D)\left\lceil \frac{N}{p^2 D} \right\rceil$. On the other hand, if $\sqrt{N} \le p \le \sqrt{a_N}$, the number of terms divisible by p^2 is at most 1, full stop. And if $p > \sqrt{a_N}$, then p^2 can't divide any terms. Therefore, the number of not-squarefree good terms is bounded by

$$\begin{aligned} &\sum_{p<\sqrt{N}} \left(\varphi(D) \cdot \left\lceil \frac{N}{p^2 D} \right\rceil\right) + \sum_{\sqrt{N} \le p \le \sqrt{a_N}} 1 \\ &< \frac{\varphi(D)}{D} N \sum_p \frac{1}{p^2} + \left(\varphi(D) \cdot O\left(\frac{\sqrt{N}}{\log N}\right) + O\left(\frac{\sqrt{ND}}{\log(ND)}\right)\right) \\ &< \frac{\varphi(D)}{2D} N + \varphi(D) \cdot O\left(\frac{\sqrt{N}}{\log N}\right) \end{aligned}$$

where we have used three well-known facts: $\sum_{p \text{ prime}} p^{-2} < \frac{1}{2}$, the prime number theorem, and the inequality $0.01\sqrt{D} \le \varphi(D)$ which is valid for any $D \ge 1$ (and is proved by multiplicativity of $\varphi(x)/x$, and checking at prime powers).

On the other hand, the number of good terms was at least

$$\varphi(D) \cdot \left\lfloor \frac{N}{D} \right\rfloor > \frac{\varphi(D)}{D} N - \varphi(D)$$

So we will have a squarefree term as long as

$$\frac{\varphi(D)}{2D} N > \varphi(D) \cdot O\left(\frac{\sqrt{N}}{\log N}\right)$$

which is certainly true if $N > O(\sqrt{D})$. As $D < a_2$, we're done.