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Twitch Solves ISL

Episode 109

Problem

Don Miguel places a token in one of the $(n+1)^2$ vertices determined by an $n \times n$ board. A move consists of moving the token from the vertex on which it is placed to an adjacent vertex which is at most $\sqrt{2}$ away, as long as it stays on the board. A path is a sequence of moves such that the token was in each one of the $(n+1)^2$ vertices exactly once. What is the maximum number of diagonal moves (those of length $\sqrt{2}$) that a path can have in total?

Video

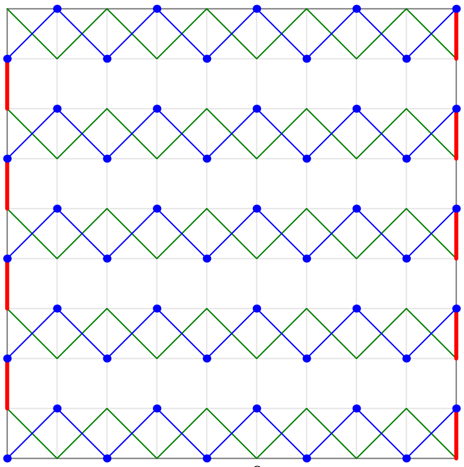
https://youtu.be/5Q7Qg3Oti-g

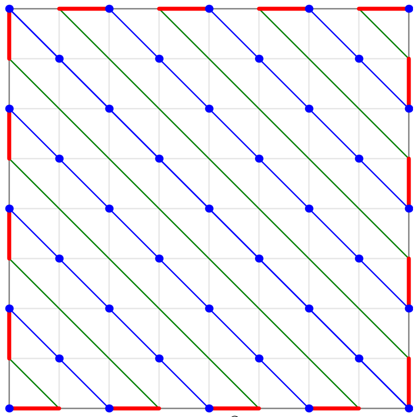
External Link

https://aops.com/community/p13136470

Solution

When n is odd, the answer is n^2+n, while when n is even, the answer is n^2. Constructions are shown below for $n=9$ and $n=8$ which obviously generalize.


$n=9$


$n=8$

To show the bound, impose coordinates (x,y) with $0 \le x \le n$ and $0 \le y \le n$, and color blue any point with $x+y \equiv 0 \pmod 2$ (as shown in the examples above); else color it green (not drawn). Imagine taking our given path and deleting every orthogonal move of distance 1 (i.e. not $\sqrt{2}$). Doing this deletion gives us a bunch of *zigzags* — graph-theoretic paths of length 1 or more where every two adjacent vertices are $\sqrt{2}$ apart. The zigzags will alternate between passing through blue dots (we call these *blue zigzags*) and through green dots (we call these *green zigzags*).

Case where n is even We give an estimate on the number of blue zigzags.

Claim. If n is even, there are at least $n+1$ blue zigzags.

Proof. There are $\left(\frac{1}{2}n+1\right)^2$ blue dots in odd-numbered rows, and $\left(\frac{1}{2}n\right)^2$ blue dots in even-numbered rows. The difference between this is $n+1$. Since blue zigzags must alternate between odd-numbered rows and even-numbered rows, there must be at least $n+1$. $\square$

Since we know there are at least $n+1$ blue zigzags, there must have been at least n green zigzags, so there were a total of at least $2n+1$ zigzags. In other words, there were at least $2n$ orthogonal moves in the original given path. So the number of diagonal steps is at most $((n+1)^2-1)-2n = n^2$.

Case where n is odd The blue and green zigzags play the same role, so we bound both:

Claim (USAMO 2008/3)**.** If n is odd, there are at least $\frac{n+1}{2}$ blue zigzags, and at least $\frac{n+1}{2}$ green zigzags.

Proof. This is USAMO 2008 problem 3 rotated by 90°. $\square$

Hence there were a total of at least $n+1$ zigzags. In other words, there were at least n orthogonal moves in the original given path. So the number of diagonal steps is at most $((n+1)^2-1)-(n+1) = n^2+n$.