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Twitch Solves ISL

Episode 109

Problem

Determine all polynomials $P(x)$ with degree $n \geq 1$ and integer coefficients so that for every real number x,

$$P(x) = (x - P(0))(x - P(1))(x - P(2)) \cdots (x - P(n-1)).$$

Video

https://youtu.be/_91OtUwJMjQ

External Link

https://aops.com/community/p13131585

Solution

The answer is $P(x) = x$ only.

For $n = 1$ it's easy to check that $P(x) = x$ is the only solution. We prove there are no others.

Claim. If $n > 1$, any potential solution would need to have $P(0) = 0$.

Proof. If not, plug in $x = 0$ and cancel $P(0)$ to get

$$1 = (-1)^n P(1)P(2)\dots P(n-1).$$

So we'd need to have $P(k) = \pm 1$ for $k = 1, \dots, n-1$. However, $P(1) \neq 0$, so actually $P(k) = -1$ for $k = 1, \dots, n-1$. In other words $P(x) = x(x+1)^{n-1}$, which doesn't work for $n \geq 2$. □

We manually bash $n = 2$ now: if $P(x) = x^2 + bx$ then we'd need $P(x) = x^2 + bx = (x-0)(x-(1+b))$, which is impossible.

Now assume $n \geq 3$.

Claim. For $k = 3, 4, \dots, n-1$, any potential solution would have $P(k) = 0$.

Proof. Note that by plugging in k, we get

$$P(k) = k \cdot (k - P(k)) \cdot T_k$$

for some integer T_k (namely $T_k = \prod_{\substack{1 \leq i \leq n-1 \\ i \neq k}} (k - P(i))$, but we don't need the actual value). This rearranges to

$$(1 + kT_k)P(k) = k^2 T_k.$$

However, $1 + kT_k$ shares no factors with $k^2 T_k$. So this can only occur if $1 + kT_k \in \{-1, 1\}$ or $T_k = 0$. For $k \geq 3$, we extract $P(k) = 0$. □

Now, we are left with $P(x) = x(x - P(1))(x - P(2))x^{n-3}$. So plug in $x = 1$ and $x = 2$:

$$\begin{aligned} P(1) &= (1 - P(1))(1 - P(2)) \\ P(2) &= 2(2 - P(1))(2 - P(2)) \cdot 2^{n-3}. \end{aligned}$$

Solve to get $P(1) = \frac{2(2^n-2)}{2^n-8}$ and $P(2) = \frac{3 \cdot 2^n}{2^n+4}$. This means $n \neq 3$, and for $n > 3$ we have $2^n + 4 \nmid 3 \cdot 2^n$ (since $2^n + 4 \nmid -12$), so there are no other solutions.