# Iberoamerican 2019/2 <br> Evan Chen 

## Twitch Solves ISL

Episode 109

## Problem

Determine all polynomials $P(x)$ with degree $n \geq 1$ and integer coefficients so that for every real number $x$,

$$
P(x)=(x-P(0))(x-P(1))(x-P(2)) \cdots(x-P(n-1))
$$

## Video

https://youtu.be/_910tUwJMjQ

## External Link

https://aops.com/community/p13131585

## Solution

The answer is $P(x)=x$ only.
For $n=1$ it's easy to check that $P(x)=x$ is the only solution. We prove there are no others.

Claim. If $n>1$, any potential solution would need to have $P(0)=0$.
Proof. If not, plug in $x=0$ and cancel $P(0)$ to get

$$
1=(-1)^{n} P(1) P(2) \ldots P(n-1)
$$

So we'd need to have $P(k)= \pm 1$ for $k=1, \ldots, n-1$. However, $P(1) \neq 0$, so actually $P(k)=-1$ for $k=1, \ldots, n-1$. In other words $P(x)=x(x+1)^{n-1}$, which doesn't work for $n \geq 2$.

We manually bash $n=2$ now: if $P(x)=x^{2}+b x$ then we'd need $P(x)=x^{2}+b x=$ $(x-0)(x-(1+b))$, which is impossible.

Now assume $n \geq 3$.
Claim. For $k=3,4, \ldots, n-1$, any potential solution would have $P(k)=0$.
Proof. Note that by plugging in $k$, we get

$$
P(k)=k \cdot(k-P(k)) \cdot T_{k}
$$

for some integer $T_{k}$ (namely $T_{k}=\prod_{\substack{1 \leq i \leq n-1 \\ i \neq k}}(k-P(i))$, but we don't need the actual value). This rearranges to

$$
\left(1+k T_{k}\right) P(k)=k^{2} T_{k}
$$

However, $1+k T_{k}$ shares no factors with $k^{2} T_{k}$. So this can only occur if $1+k T_{k} \in\{-1,1\}$ or $T_{k}=0$. For $k \geq 3$, we extract $P(k)=0$.

Now, we are left with $P(x)=x(x-P(1))(x-P(2)) x^{n-3}$. So plug in $x=1$ and $x=2$ :

$$
\begin{aligned}
& P(1)=(1-P(1))(1-P(2)) \\
& P(2)=2(2-P(1))(2-P(2)) \cdot 2^{n-3}
\end{aligned}
$$

Solve to get $P(1)=\frac{2\left(2^{n}-2\right)}{2^{n}-8}$ and $P(2)=\frac{3 \cdot 2^{n}}{2^{n}+4}$. This means $n \neq 3$, and for $n>3$ we have $2^{n}+4 \nmid 3 \cdot 2^{n}$ (since $2^{n}+4 \nmid-12$ ), so there are no other solutions.

