All-Lincoln 2022/6

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Twitch Solves ISL

Episode 109

Problem

Consider acute triangle ABC. Let D, E, F be the A, B, C intouch points of ABC, and X, Y, Z as the arc midpoints of BC, CA, AB in the circumcircle of ABC. Prove that the triangle bounded by the lines XE, YF, ZD has area at most half of the area of ABC.

Video

https://youtu.be/GEa2nOS1PBM

Solution

The following stronger claim is true:

Claim. Let DEF be any triangle. Let XYZ be a triangle obtained from a homothety of ratio $\rho \geq 1$ whose center lies inside $\triangle DEF$. Then the triangle bounded by the lines XE, YF, ZD has area at most ρ of the area of DEF.

Proof. Brute-force bary on $\triangle DEF$. Let $\lambda = \rho - 1 \geq 0$, and $\mu = \lambda^{-1}$. Also, let the homothety center be (u, v, w) for $u, v, w > 0$ and $u + v + w = 1$. Then

$$\begin{aligned} X &= (\lambda(v+w)+1, -\lambda v, -\lambda w) . \\ &= (v + w + \mu : -v : -w) \\ Y &= (-u : w + u + \mu : -w) \\ Z &= (-u : -v : u + v + \mu) \\ DZ \cap EX &= ((u+v+\mu)(v+w+\mu) : wv : -w(u+v+\mu)) \\ EX \cap FY &= (-u(v+w+\mu) : (v+w+\mu)(w+u+\mu) : uw) \\ FY \cap DZ &= (uv : -v(w+u+\mu) : (w+u+\mu)(u+v+\mu)) . \end{aligned}$$

Direct computation gives that

$$\frac{\operatorname{Area}(DZ \cap EX, EX \cap FY, FY \cap DZ)}{[DEF]} = \frac{(uvw + \prod_{\text{cyc}}(u+v+\mu))^2}{\prod_{\text{cyc}}(\mu^2 + (u+2v)\mu + v(u+v+w))}.$$

Therefore, since $\rho = \mu^{-1} + 1$, we need to show

$$\begin{aligned} \left(uvw + \prod_{\text{cyc}}(u+v+\mu)\right)^2 \leq \left(1 + \frac{u+v+w}{\mu}\right) \\ \cdot \prod_{\text{cyc}} \left(\mu^2 + (u+2v)\mu + v(u+v+w)\right) . \end{aligned}$$

However, using Sage reveals that

$$\begin{aligned} \text{RHS} - \text{LHS} = \mu^4&(uv + vw + wu) \\ &+ \mu^3 \left(4(uv^2 + vw^2 + wu^2) + 2(u^2v + v^2w + w^2u) + 9uvw\right) \\ &+ \mu^2 \sum_{\text{cyc}} (u^3v + 6u^2v^2 + 6uv^3 + 20u^2vw) \\ &+ \mu \sum_{\text{cyc}} (2u^3v^2 + 4uv^4 + 6u^2v^3 + 19u^3vw + 32u^2v^2w) \\ &+ \sum_{\text{cyc}} (u^3v^3 + u^5w + 2u^2v^4 + 8u^4vw + 19u^2vw^3 + 20u^2v^3w + 12u^2v^2w^2) \\ &+ \frac{1}{\mu} \sum_{\text{cyc}} \left(u^5vw + 4u^2v^4w + 4u^2vw^4 + 6u^3v^3w + 12u^2v^2w^3\right) \\ &\geq 0. \end{aligned}$$

□

Remark. Note that equality occurs if say $D = X$, which corresponds to $v = w = 0$.

We now use the following theorem.

Theorem (Apparently not well-known)**.** We have $[DEF]/[ABC] = 2R/r$, where r and R are the inradius and circumradius.

Note that in the original problem, $\triangle DEF$ and $\triangle XYZ$ are homothetic with ratio $\frac{YZ}{EF} = \frac{R}{r}$. Their homothety center is the concurrence point X_{56} of lines DX, EY and FZ, so we'd be done upon showing:

Claim (Annoying interior analysis)**.** When $\triangle ABC$ is acute, X_{56} lies inside $\triangle DEF$.

Proof. Let I denote the incenter, so I is the orthocenter of acute triangle XYZ and in particular lies inside acute triangle DEF. Then $\overline{YZ}$ is the perpendicular bisector of $\overline{AI}$, while $\overline{EF}$ is perpendicular to $\overline{AI}$ at a point closer to I than A (because $\angle A < 90^\circ \implies \angle EIF > 90^\circ$). Hence $F = \overline{EF} \cap \overline{DF}$ lies inside $\triangle XYZ$, and so $\overline{ZF}$ is an internal cevian of $\triangle XYZ$. The same is true for $\overline{DX}$ and $\overline{EY}$, and we're done. $\square$