All-Lincoln 2022/6

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Twitch Solves ISL

Episode 109

Problem

Consider acute triangle ABC. Let D, E, F be the A, B, C intouch points of ABC, and X, Y, Z as the arc midpoints of BC, CA, AB in the circumcircle of ABC. Prove that the triangle bounded by the lines XE, YF, ZD has area at most half of the area of ABC.

Video

https://youtu.be/GEa2nOS1PBM

Solution

The following stronger claim is true:

Claim. Let DEF be any triangle. Let XYZ be a triangle obtained from a homothety of ratio $\rho \geq 1$ whose center lies inside $\triangle DEF$. Then the triangle bounded by the lines XE, YF, ZD has area at most ρ of the area of DEF.

Proof. Brute-force bary on $\triangle DEF$. Let $\lambda = \rho - 1 \geq 0$, and $\mu = \lambda^{-1}$. Also, let the homothety center be (u, v, w) for $u, v, w > 0$ and $u + v + w = 1$. Then

$$\begin{aligned}
X &= (\lambda(v+w)+1, -\lambda v, -\lambda w)\,. \\
&= (v+w+\mu : -v : -w) \\
Y &= (-u : w+u+\mu : -w) \\
Z &= (-u : -v : u+v+\mu) \\
DZ \cap EX &= ((u+v+\mu)(v+w+\mu) : wv : -w(u+v+\mu)) \\
EX \cap FY &= (-u(v+w+\mu) : (v+w+\mu)(w+u+\mu) : uw) \\
FY \cap DZ &= (uv : -v(w+u+\mu) : (w+u+\mu)(u+v+\mu))\,.
\end{aligned}$$

Direct computation gives that

$$\frac{\text{Area}(DZ \cap EX, EX \cap FY, FY \cap DZ)}{[DEF]} = \frac{(uvw + \prod_{\text{cyc}}(u+v+\mu))^2}{\prod_{\text{cyc}}(\mu^2 + (u+2v)\mu + v(u+v+w))}.$$

Therefore, since $\rho = \mu^{-1} + 1$, we need to show

$$\begin{aligned}
\left(uvw + \prod_{\text{cyc}}(u+v+\mu)\right)^2 \leq \left(1 + \frac{u+v+w}{\mu}\right) \\
\cdot \prod_{\text{cyc}} \left(\mu^2 + (u+2v)\mu + v(u+v+w)\right).
\end{aligned}$$

However, using Sage reveals that

$$\begin{aligned}
\text{RHS} - \text{LHS} &= \mu^4(uv + vw + wu) \\
&+ \mu^3 \left(4(uv^2 + vw^2 + wu^2) + 2(u^2v + v^2w + w^2u) + 9uvw\right) \\
&+ \mu^2 \sum_{\text{cyc}} (u^3v + 6u^2v^2 + 6uv^3 + 20u^2vw) \\
&+ \mu \sum_{\text{cyc}} (2u^3v^2 + 4uv^4 + 6u^2v^3 + 19u^3vw + 32u^2v^2w) \\
&+ \sum_{\text{cyc}} (u^3v^3 + u^5w + 2u^2v^4 + 8u^4vw + 19u^2vw^3 + 20u^2v^3w + 12u^2v^2w^2) \\
&+ \frac{1}{\mu} \sum_{\text{cyc}} \left(u^5vw + 4u^2v^4w + 4u^2vw^4 + 6u^3v^3w + 12u^2v^2w^3\right) \\
&\geq 0.
\end{aligned}$$

□

Remark. Note that equality occurs if say $D = X$, which corresponds to $v = w = 0$.

Less terrible proof of the claim, sent by Darij Grinberg. Let P be the center of the homothety, and let $U = DZ \cap EX$ and $V = FY \cap DZ$ and $W = EX \cap FY$. We must show that $[UVW] \leq \rho[DEF]$.

The line FP intersects both (closed) segments EP and ED, so it also intersects the closed segment EX (since X is on the segment DP). In other words, W lies on the segment EX. On the other hand, the point U lies on the extension of this segment beyond X, since it lies between X and $EU \cap FD$ (because the point Z lies between P and F). Hence, the point X lies on the segment WU. Similarly, Y lies on the segment VW, and Z lies on the segment UV. As P lies inside $\triangle XYZ$,

$$[UVW] = [PXWY] + [PYVZ] + [PZUX].$$

But since W and Y lie on the segments EU and EP, we have

$$[PXWY] \le [PXE] = \rho[PDE]$$

and similarly (cyclically)

$$[PYVZ] \le \rho[PEF], \qquad [PZUX] \le \rho[PFD].$$

Summing these three inequalities yields

$$[PXWY] + [PYVZ] + [PZUX] \le \rho[PDE] + \rho[PEF] + \rho[PFD] = \rho[DEF]$$

as desired. □

We now use the following theorem.

Theorem (Apparently not well-known)**.** We have $[DEF]/[ABC] = 2R/r$, where r and R are the inradius and circumradius.

(This is possibly a bit overkill, as all that's needed is $R/r \ge 2$ here.)

Note that in the original problem, $\triangle DEF$ and $\triangle XYZ$ are homothetic with ratio $\frac{YZ}{EF} = \frac{R}{r}$. Their homothety center is the concurrence point X_{56} of lines DX, EY and FZ, so we'd be done upon showing:

Claim (Annoying interior analysis)**.** When $\triangle ABC$ is acute, X_{56} lies inside $\triangle DEF$.

Proof. Let I denote the incenter, so I is the orthocenter of acute triangle XYZ and in particular lies inside acute triangle DEF. Then $\overline{YZ}$ is the perpendicular bisector of $\overline{AI}$, while $\overline{EF}$ is perpendicular to $\overline{AI}$ at a point closer to I than A (because $\angle A < 90^\circ \implies \angle EIF > 90^\circ$). Hence $F = \overline{EF} \cap \overline{DF}$ lies inside $\triangle XYZ$, and so $\overline{ZF}$ is an internal cevian of $\triangle XYZ$. The same is true for $\overline{DX}$ and $\overline{EY}$, and we're done. □