Putnam 2022 B4

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Twitch Solves ISL

Episode 108

Problem

Find all integers n with $n \geq 4$ for which there exists a sequence of distinct real numbers $x_1, \ldots, x_n$ such that each of the sets

$$\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \ldots, \{x_{n-2}, x_{n-1}, x_n\}, \{x_{n-1}, x_n, x_1\}, \text{ and } \{x_n, x_1, x_2\}$$

forms a 3-term arithmetic progression when arranged in increasing order.

Video

https://youtu.be/C9WhZldyMuc

External Link

https://aops.com/community/p26632989

Solution

The answer is $3 \mid n$.

Proof of necessity. By scaling we may assume WLOG that $x_1 = 0$ and $x_2 = 1$. Now if $\{a, b, c\}$ is an arithmetic progression in some order, then in general

$$c \in \left\{ 2b - a, 2a - b, \frac{a+b}{2} \right\}$$

and so in any valid sequence of x_i's, we find that every x_i is a rational number whose denominator is a power of 2 (by induction, starting from x_1 and x_2 and propagating to x_3, x_4, ...).

This means it's possible to take modulo 3.

When we do this we find every triple consists of $0 \pmod 3$, $1 \pmod 3$, $2 \pmod 3$ in some order; more precisely induction gives

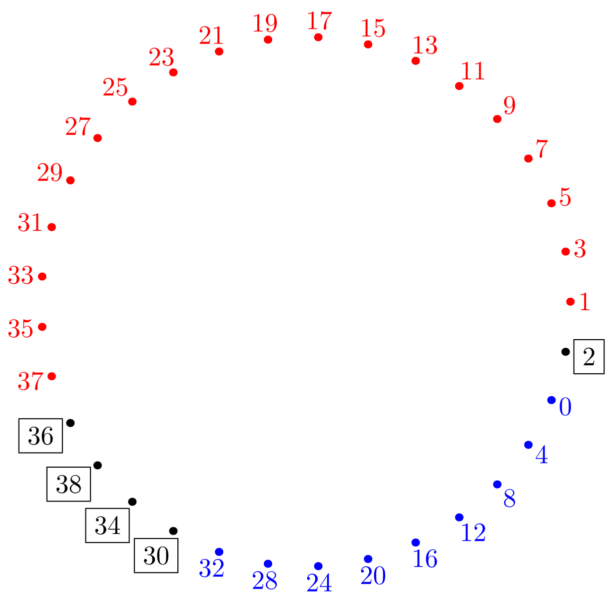
$$x_i = \begin{cases} 1 \mod 3 & i \equiv 1 \mod 3 \\ 2 \mod 3 & i \equiv 2 \mod 3 \\ 0 \mod 3 & i \equiv 0 \mod 3. \end{cases}$$

Since indices wrap modulo n, this forces $3 \mid n$.

Construction. For $k \geq 1$, we give a construction valid for $n = (2k+1)+4+k+1 = 3k+6$:

$$(1, 3, 5, 7, \ldots, 4k-1, 4k+1, 4k, 4k+2, 4k-4, 4k-8, 4k-12, \ldots, 4, 0, 2).$$

For $k = 9$ and $n = 33$ it is illustrated below; this was first posted by `MarkBcc168` for $k = 5$ and $n = 21$.



We meanwhile use $(1, 3, 5, 4, 6, 2)$ for $n = 6$.