Math Prize 2022/2

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Twitch Solves ISL

Episode 108

Problem

Determine, with proof, whether or not there exists a *non-isosceles* trapezoid $ABCD$ such that the lengths AC and BD both lie in the set $\{DA+AB, AB+BC, BC+CD, CD+DA, AB+CD, BC+DA\}$.

Video

https://youtu.be/C9WhZldyMuc

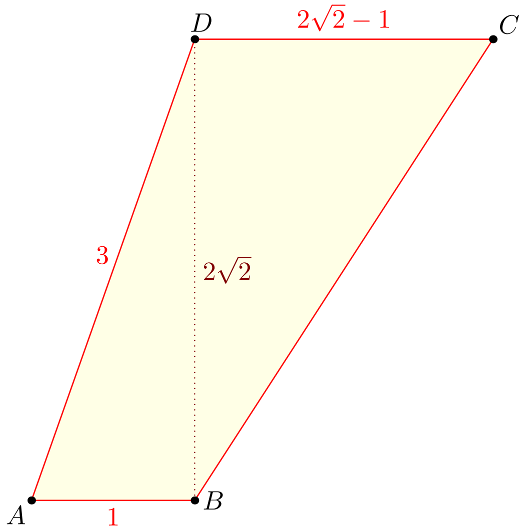
External Link

https://aops.com/community/p26628697

Solution

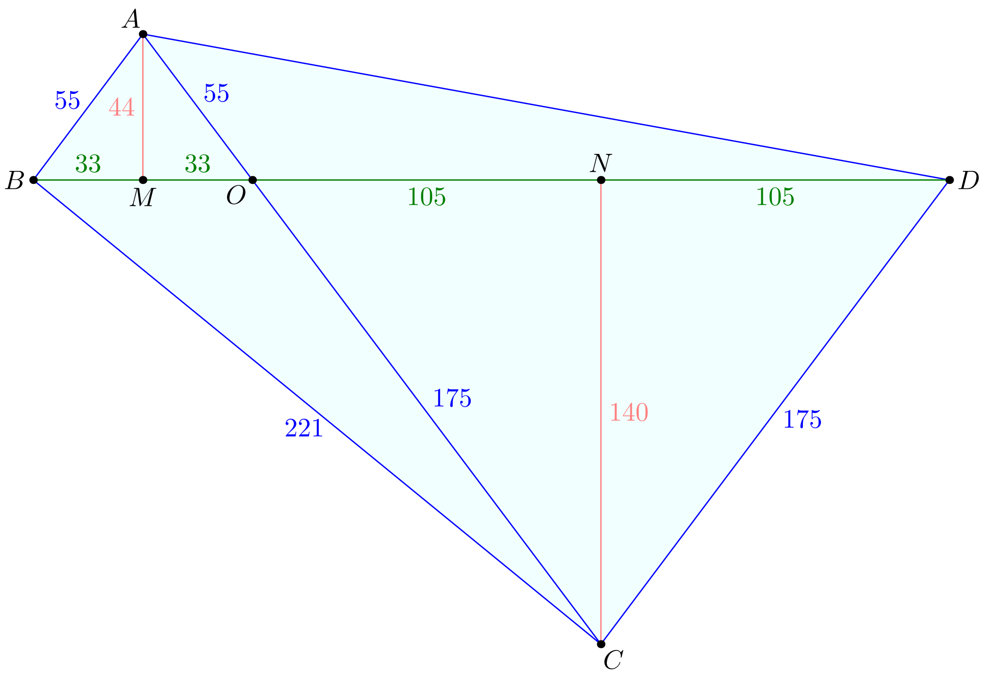
The answer is yes, such a trapezoid exists. We present two possible direct constructions and one indirect one.

First construction Let $A = (0,0)$, $B = (1,0)$, $C = (2\sqrt{2}, 2\sqrt{2})$, $D = (1, 2\sqrt{2})$, as shown below.



Then $AC = 4 = 1 + 3 = AB + AD$, and $BD = 2\sqrt{2} = 1 + (2\sqrt{2} - 1) = AB + CD$. Also, we clearly have $AB \parallel CD$ (since $\angle DBA = \angle CDB = 90^\circ$) and $3 = AD \neq BC$, so this construction is valid.

Second construction Construct two similar right triangles AOM and CON, where $OM = 33$, $MA = 44$, $OA = 55$ and $ON = 105$, $NC = 140$, $CO = 175$. Situate these triangles such that AOC and MON are collinear. Finally, let B and D be the reflections of O over M and N, respectively. The resulting figure is depicted below.



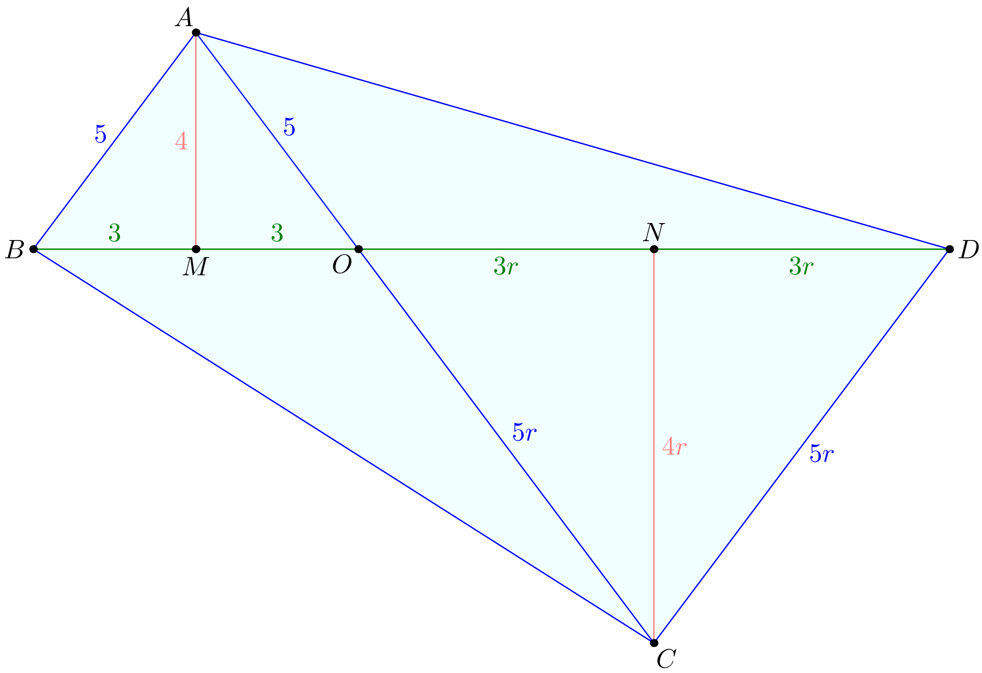
Then because $\triangle AOB$ and $\triangle COD$ are similar, it follows that AB and CD are parallel. In that case, we have

$$\begin{aligned} AB &= AO = 55 \\ CD &= CO = 175 \\ BC &= \sqrt{BN^2 + NC^2} = \sqrt{171^2 + 140^2} = 221 \\ AC &= AO + CO = 55 + 175 = 230 = AB + CD \\ BD &= BO + DO = 2(33 + 105) = 276 = BC + CD. \end{aligned}$$

(The length of AD is not relevant for this solution.) This completes the construction.

Remark. The numbers selected here may seem magical in nature. Really, the underlying idea is to construct two similar isosceles triangles AOB and COD as above, so that $AB \parallel CD$ and $AC = AB + CD$ are automatically true. In that case, the only condition that needs to hold is for $BD = AB + BC$ to be true. Because we have a choice of three numbers (the length AO, BO, CO determine the figure), it should be possible to make this equation true, and one simply needs to exhibit one solution. The lengths here were chosen after some calculation to yield a construction in which all lengths were integers, but this is neither necessary nor important for the solution to work.

Indirect construction using continuity We develop the ideas mentioned in the preceding remark by showing how one can indirectly prove the existence of a valid trapezoid, without having to actually find all the necessary constants. Indeed, we again construct two similar right triangles AOM and CON, but this time we set where $OM = 3$, $MA = 4$, $OA = 5$ (say) and $ON = 3r$, $NC = 4r$, $CO = 5r$, for some $r > 0$. Then let B and D be the reflections of O over M and N, respectively.



Because $BD = 6 + 6r$ and $AC = 5 + 5r$, we have $BD > AC$, and this trapezoid is not isosceles for any value of r. Now, we vary the parameter r and consider the function

$$f(r) := AB + BC - BD = 5 + \sqrt{(3r+6)^2 + (4r)^2} - (6 + 6r).$$

Note that this function is continuous and

$$\begin{aligned} f(0.001) &= 5 + \sqrt{3.006^2 + 4^2} - 6.006 > 0 \\ f(1000) &= 5 + \sqrt{3006^2 + 4000^2} - 6006 < 0. \end{aligned}$$

so by the intermediate value theorem, there must be some $0.001 < r < 1000$ for which $f(r) = 0$. That value of r gives a valid construction.

Remark. As we saw in the previous solution, $r = \frac{35}{11}$ works. In fact, it is the unique value of r for which $f(r) = 0$.

The choice of a 3-4-5 triangle in this construction is just for concreteness; many other dimensions would work as well.