DEMO Mock 2022/1

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Twitch Solves ISL

Episode 107

Problem

Determine all positive integers N where the equations

$$ab - cd = a + b + c + d = N$$

have at least one solution for positive integers a, b, c, and d.

Video

https://youtu.be/XUAVeDk2GsY

External Link

https://aops.com/community/p25441644

Solution

Ignore N for a moment and write the equation as $(a-1)(b-1) = (c+1)(d+1)$. Then the factor lemma shows that it's equivalent to

$$\underbrace{(a-1)}_{pq}\underbrace{(b-1)}_{rs} = \underbrace{(c+1)}_{pr}\underbrace{(d+1)}_{qs}$$

So now $N = a+b+c+d = pq+rs+pr+qs = (p+s)(r+q)$.

The only caveat is we can't find $p = r = 1$ or $q = s = 1$. So we get that any composite N other than $N = 4$ or $N = 6$ is OK.