Brazil 2022/6

Evan Chen

TWITCH SOLVES ISL

Episode 107

Problem

Some cells of a 10×10 grid are colored blue. A set of six cells is called *gremista* when the cells are the intersection of three rows and two columns, or two rows and three columns, and are painted blue. Determine the greatest value of n for which it is possible to color n chessboard cells blue such that there is not a gremista set.

Video

https://youtu.be/oxrY6brGYfg

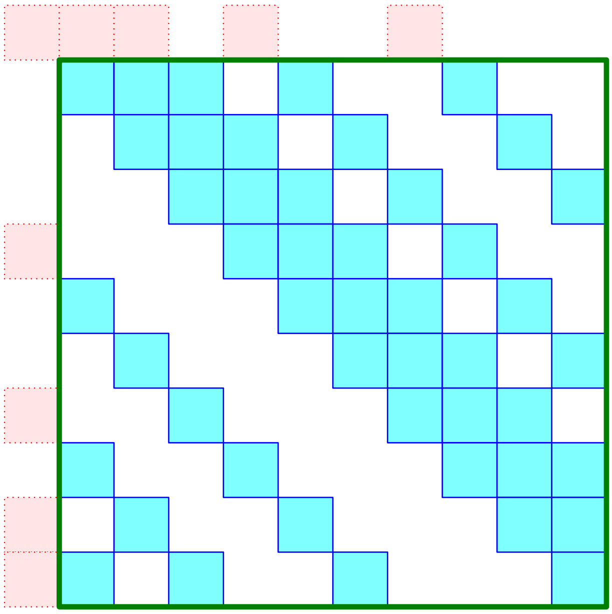
External Link

https://aops.com/community/p26563956

Solution

The answer is 46. Call a grid *valid* if no gremista cells exist.

Construction. The construction is shown below in the thick green border, which encloses a valid construction of a 10×10.



The reason for the "extra" row and column is that this construction is actually obtained by taking an 11×11 grid and using cells $\{y, y+1, y+2, y+4, y+7\} \pmod{11}$ in the y'th row; this gives a valid 11×11 grid, and deleting a single row and column of a valid 11×11 will always leave a valid 10×10.

Bound. We start with:

Claim. There must be at most 47 blue cells in a valid 10×10 grid. Also, if such a valid grid did exist, then there exists seven columns with at least 35 blue cells among them.

Proof. Let a_i be the number of blue cells in the ith column. Then the usual double-counting argument gives

$$\sum_1^{10} \binom{a_i}{2} \le 2\binom{10}{2} = 90.$$

Because $\binom{\bullet}{2}$ is convex, for a fixed value of $\sum a_i$ the left-hand side is minimized when any two a_i differ by at most one. The unique choices of a_i (up to permutation) for which $\sum a_i = 47$ and $\sum \binom{a_i}{2} < 90$ are $3\binom{4}{2} + 7\binom{5}{2} = 88$ and $4\binom{4}{2} + 5\binom{5}{2} + \binom{6}{2} = 89$; no such choices exist with $\sum a_i \ge 48$. □

But now we can repeat the same convexity argument on just those seven columns; assuming for contradiction an example for 47 did exist, let b_i denote the number of blue cells in the ith row *and* in those seven columns. From $\sum b_i \ge 35$ we have

$$\sum_1^{10} \binom{b_i}{2} \le 2\binom{7}{2} = 42$$

but

$$\sum_{1}^{10} \binom{b_i}{2} > 10\binom{3.5}{2} = 43.75 > 42.$$

Another proof of bound (Jiahe Liu). Assume for contradiction there is a valid grid of 47 blue cells (if there are more than 47 cells, arbitrarily uncolor some cells). Let $a \le b \le c$ be the counts of blue cells in the rows with the fewest blue cells (ties broken arbitrarily).

Claim. We must have $a + b + c \le 12$.

Proof. If $c \le 4$, the result is clear. When $c \ge 5$, the other seven rows need to have at least c cells, so we have

$$47 \ge a + b + 8c \implies a + b + c \le 47 - 7c \le 12. \qquad \square$$

Thus the remaining seven rows have at least 35 blue cells among them. We can now finish as in the first solution.