Brazil 2022/6

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Twitch Solves ISL

Episode 107

Problem

Some cells of a 10×10 are colored blue. A set of six cells is called *gremista* when the cells are the intersection of three rows and two columns, or two rows and three columns, and are painted blue. Determine the greatest value of n for which it is possible to color n chessboard cells blue such that there is not a gremista set.

Video

https://youtu.be/oxrY6brGYfg

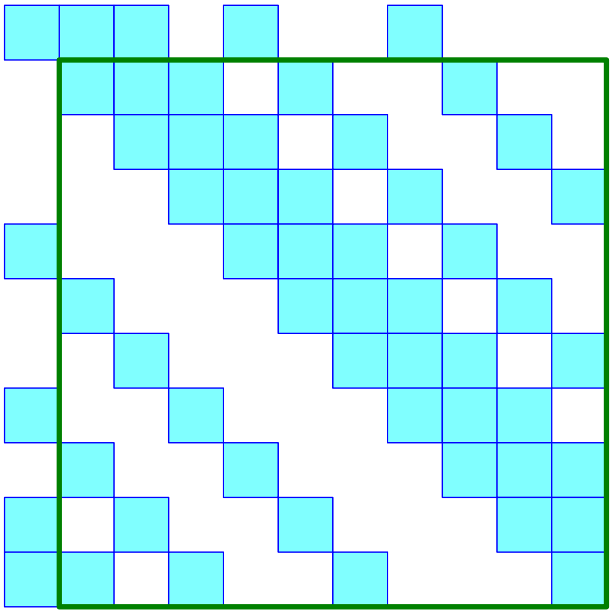
External Link

https://aops.com/community/p26563956

Solution

The answer is 46. Call a grid *valid* if no gremista cells exist.

Construction The construction is shown below in the thick green border, which encloses a valid construction of a 10×10.



The reason for the "extra" row and column is that this construction is actually obtained by taking an 11×11 grid and using cells $\{y, y+1, y+2, y+4, y+7\} \pmod{11}$ in the y'th row; this gives a valid 11×11 grid, and deleting a single row and column of a valid 11×11 will always leave a valid 10×10.

Bound We start with:

Claim. There must be at 47 blue cells in an 10×10 valid grid. Also, if such a valid grid did exist, it would have to have exactly seven columns with exactly five blue cells each.

Proof. Let a_i be the number of blue cells in the ith column. Then the usual double-counting argument gives

$$\sum_1^{10} \binom{a_i}{2} \le 2\binom{10}{2} = 90.$$

Because $\binom{\bullet}{2}$ is convex, for a fixed value of $\sum a_i$ the left-hand side is minimized when any two a_i differ by at most one. So we find that the unique choice of a_i (up to permutation) for which $\sum a_i = 47$ and $\sum \binom{a_i}{2} < 90$ is $3\binom{4}{2} + 7\binom{5}{2} = 88$; no such choices exist with $\sum a_i \ge 48$. $\square$

But now we can repeat the same convexity argument on just those seven columns; assuming for contradiction an example for 47 did exist, let b_i denote the number of blue cells in the ith row *and* in those seven columns. Then we have $\sum b_i = 7 \cdot 5 = 35$ yet

$$\sum_1^{10} \binom{b_i}{2} \le 2\binom{7}{2} = 42$$

but

$$\sum_{1}^{10} \binom{b_i}{2} > 10\binom{3.5}{2} = 43.75 > 42.$$