Australia 2020/8

Evan Chen

Twitch Solves ISL

Episode 107

Problem

Prove that for each integer k satisfying $2 \le k \le 100$, there are positive integers b_2, b_3, $\dots$, b_{101} such that

$$b_2^2 + b_3^3 + \dots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \dots + b_{101}^{101}.$$

Video

https://youtu.be/Qqk8cbL2fP8

Solution

For an integer M to be chosen later, we will choose

$$b_2^2 = 69696 M^{100!}$$
$$b_3^3 = \cdots = b_{100}^{100} = M^{100!}.$$

(Note that $69696 = 264^2$.) Then the desired equation becomes

$$b_{101}^{101} = (69696 + (k-2) - (100-k)) \cdot M^{100!}$$

and so we can let $M = 69696 + (k-2) - (100-k)$ and we're OK, since $M^{100!+1}$ is obviously a 100th power by Wilson's theorem.