Twitch 106.1

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Twitch Solves ISL

Episode 106

Problem

Given an equilateral triangle ABC and a point X, let the reflection of X about AB be X_C, and the other reflections defined similarly. Prove or disprove whether or not AX_A, BX_B, and CX_C are concurrent or parallel.

Video

https://youtu.be/9iE3J43Sozk

Solution

Use complex numbers with (ABC) the unit circle. Ignore for now the condition that ABC is equilateral to retain symmetry. We know that

$$x_A = b + c - bc\overline{x}.$$

The equation $Z \in AX_A$ is then given by

$$0 = \det \begin{bmatrix} a & \overline{a} & 1 \\ b + c - bc\overline{x} & \overline{b} + \overline{c} - \overline{b}\overline{c}x & 1 \\ z & \overline{z} & 1 \end{bmatrix}$$

or

$$(\overline{a}b + \overline{a}c - \overline{a}bc\overline{x}) - (a\overline{b} + a\overline{c} - a\overline{b}\overline{c}x) = (\overline{a} - \overline{b} - \overline{c} + \overline{b}\overline{c}x)z - (a - b - c + bc\overline{x})\overline{z}$$

Consider the analogous equations for BX_B and CX_C to get a 3×3 determinant.

Claim. When $a = 1$, $b = e^{\frac{2}{3}\pi i}$, $c = e^{\frac{4}{3}\pi i}$, this determinant vanishes. In fact, the sum of the rows is the zero vector.

Proof. Note that $abc = 1$, $a^{-1} + b^{-1} + c^{-1} = a + b + c = a^2 + b^2 + c^2 = 0$.

$$\begin{aligned}
\sum_{\text{cyc}} \overline{a}b + \overline{a}c &= \sum_{\text{cyc}} \overline{a}(-a) = -3 \\
\sum_{\text{cyc}} \overline{a}bc\overline{x} &= \sum_{\text{cyc}} a^{-2}\overline{x} = 0 \\
\sum_{\text{cyc}} a\overline{b}\overline{c}x &= \sum_{\text{cyc}} a^2 x = 0 \\
\sum_{\text{cyc}} (\overline{a} - \overline{b} - \overline{c}) &= \sum_{\text{cyc}} 2\overline{a} = 0 \\
\sum_{\text{cyc}} (a - b - c) &= \sum_{\text{cyc}} 2a = 0 \\
\sum_{\text{cyc}} (\overline{b}\overline{c})x &= \sum_{\text{cyc}} ax = 0 \\
\sum_{\text{cyc}} bc\overline{x} &= \sum_{\text{cyc}} a^{-1}x = 0.
\end{aligned}$$

□