ToT Spring 2004 S-A5

Evan Chen

Twitch Solves ISL

Episode 106

Problem

A circle and a parabola share exactly two points, A and B. Suppose they are tangent at A (that is, the tangent lines to the circle and parabola at A coincide). Does it follow that they are also tangent at B?

Video

https://youtu.be/tAXEpNhZG_Q

External Link

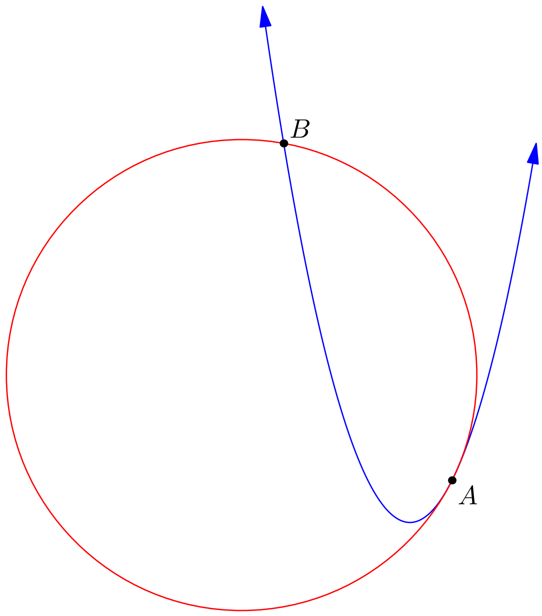
https://aops.com/community/p14147724

Solution

It does not follow; for a counterexample use the construction

$$y = x^2 \qquad \text{and} \qquad (x+4)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{125}{4}.$$

See figure below. They meet at $A = (1,1)$ and $B = (-3,9)$.



Remark (Motivation for these numbers)**.** If we write the parabola as $y = x^2$ WLOG, and the circle as $(x-\alpha)^2 + (y-\beta)^2 = \gamma^2$, then the intersections are controlled by a quartic in x, namely:

$$\begin{aligned} 0 &= (x-\alpha)^2 + \left(x^2 - \beta\right)^2 - \gamma^2 \\ &= x^4 + (1-2\beta)x^2 - 2\alpha x + (\alpha^2 + \beta^2 - \gamma^2). \end{aligned}$$

To fail the problem, we need this quartic to have a "triple root" at A. The above example is constructed by choosing $(x-1)^3(x+3)$ as the quartic and equating coefficients.