Kosovo TST 2019/1

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Twitch Solves ISL

Episode 106

Problem

There are 2019 cards in a box. Each card has a number written on one of its sides and a letter on the other side. Amy and Ben play the following game: in the beginning Amy takes all the cards, places them on a line and then she flips as many cards as she wishes. Each time Ben touches a card he has to flip it and its neighboring cards. Ben is allowed to have as many as 2019 touches. Ben wins if all the cards are on the numbers' side, otherwise Amy wins. Determine who has a winning strategy.

Video

https://youtu.be/1HWZAbOob44

External Link

https://aops.com/community/p12173180

Solution

Ben wins.

First, note: 2019 doesn't matter because moves commute with each other and repeating a move does nothing. So we're going to ignore the requirement of at most 2019 touches.

We now give an algorithm to flip any individual single card, which is obviously sufficient. Number the cards 1, ..., 2019 from left to right.

- Touch cards 1, 2, 4, 5, 7, 8, ..., 2017, 2018 will flip only the card 2019.
- Touching 2019 in addition to the previous algorithm will flip only the card 2018.
- If we do the algorithm to flip just n and $n-1$, and then touch $n-1$, that's the same as flipping just $n-2$.
- Repeating this way, we get an algorithm to flip any individual card.

Remark. In linear algebra terms, the problem can also be phrased as saying that

$$\det \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \end{bmatrix} \neq 0$$

in $\mathbb{F}_2$.