Brazil 2022/2

Evan Chen

Twitch Solves ISL

Episode 106

Problem

Let ABC be an acute triangle, with $AB < AC$. Let K be the midpoint of the arc BC that does not contain A and let P be the midpoint of BC. Let I_B, I_C be the B-excenter and C-excenter of ABC, respectively. Let Q be the reflection of K with respect to A. Prove that the points P, Q, I_B, I_C are concyclic.

Video

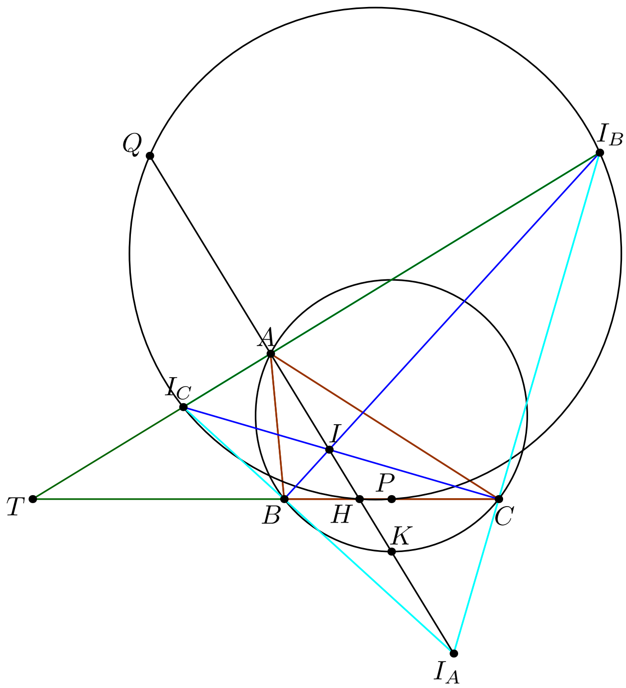
https://youtu.be/w0HvkqfOaQ8

External Link

https://aops.com/community/c6h2965497p26563619

Solution

Let I_A be the A-excenter, and let $H = \overline{AIK} \cap \overline{BC}$.



- Then by Brokard's theorem on cyclic quadrilateral $BICI_A$, it follows that $\triangle I_B I_C H$ is self-polar to this circle.
- In particular K is the orthocenter. Equivalently, H is the orthocenter of $\triangle I_B K I_C$.
- Define $T = \overline{I_B I_C} \cap \overline{BC}$. Then $(TH; BC) = -1$.
- I claim that $I_B I_C PH$ is cyclic. Indeed, $TH \cdot TP = TB \cdot TC = TI_B \cdot I_C$.
- The reflection of the orthocenter K of $\triangle I_B I_C H$ over a side $\overline{I_B I_C}$ then coincides with Q, which now lies on $(PHI_B I_C)$.