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Episode 103

## Problem

Find all positive integers $k$ for which there exists a *nonlinear* function $f\colon \mathbb{Z} \to \mathbb{Z}$ which satisfies

$$f(a) + f(b) + f(c) = \frac{f(a-b) + f(b-c) + f(c-a)}{k}$$

for any integers $a$, $b$, $c$ with $a + b + c = 0$.

## Video

https://youtu.be/dC9VpiGVRqs

## External Link

https://aops.com/community/p26379731

## Solution

The complete set of solutions is given by

- For $k = 1$, $f(x) \equiv C_1 x + C_2 (x \mod 2) + C_3$.
- For $k = 3$, $f(x) \equiv C_1 x + C_2 x^2$.
- For $k = 9$, $f(x) \equiv C_1 x + C_2 x^4$.
- For all other $k$, only $f(x) \equiv C_1 x$.

Here $C_1$, $C_2$, $C_3$ are arbitrary integers. We can check they work, so now we just want to show they are the only ones.

We will solve the functional equation for $f \colon \mathbb{Z} \to \mathbb{C}$, claiming that the above solution set remains the only one. If $k = 1$, we can shift by constants to get $f(0) = 0$; if $k \neq 1$ apply $a = b = c = 0$ to get $f(0) = 0$ anyways. Now note that $f(x) \equiv x$ is a solution, so we may shift by the identity to assume $f(-1) = f(1)$.

We will prove in this case, $f \equiv 0$ unless $k = 1, 3, 9$.

Now plug in $(a, b, c) = (n + 1, -n, -1)$ and $(a, b, c) = (1, n, -(n + 1))$ gives

$$\begin{aligned} f(2n+1) + f(-n+1) + f(-n-2) &= k\left(f(n+1) + f(-n) + f(-1)\right) \\ &= k\left(f(-n-1) + f(n) + f(1)\right). \end{aligned}$$

The last two by induction imply $f$ is even. Now, by using this and $(a, b, c) = (n, -n, 0)$ we obtain

$$\begin{aligned} f(2n+1) + f(n+2) + f(n-1) &= k\left(f(n+1) + f(n) + f(1)\right) \\ f(2n) + 2f(n) &= k\left(2f(n) + f(0)\right) \implies f(2n) = (2k-2)f(n). \end{aligned}$$

Thus $f$ is determined recursively by $f(1)$ (by induction). In particular, if $f(1) = 0$ then $f(n) \equiv 0$ by induction.

Now, let us assume $f(1) \neq 0$, and hence by scaling $f(1) = 1$. We can then compute:

$$\begin{aligned} f(1) &= 1 \\ f(2) &= 2k - 2 \\ f(3) &= k^2 \\ f(4) &= 4k^2 - 8k + 4 \\ f(5) &= k^3 - 2k^2 + 7k - 5 \\ f(6) &= 2k^3 - 2k^2 \\ f(7) &= 4k^3 - 6k^2 - 4k + 7 \\ f(8) &= 8k^3 - 24k^2 + 24k - 8. \end{aligned}$$

Plug in $(a, b, c) = (5, -3, -2)$ and compute $f(8) + f(1) + f(7) = k\left(f(5) + f(3) + f(2)\right)$, which simplifies to give

$$k^4 - 13k^3 + 39k^2 - 27k = 0 \implies k(k-1)(k-3)(k-9) = 0$$

so $k = 1$, $k = 3$, or $k = 9$. In these cases it is easy to check by induction now that $f(n) = n \mod 2$, $f(n) = n^2$, and $f(n) = n^4$.