USEMO 2022/4

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Twitch Solves ISL

Episode 103

Problem

Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points P, Q, R, S lie in the interiors of segments AB, BC, CD, DA, respectively, such that

$$\angle PDA = \angle PCB, \quad \angle QAB = \angle QDC, \quad \angle RBC = \angle RAD, \quad \text{and} \quad \angle SCD = \angle SBA.$$

Let $\overline{AQ}$ intersect $\overline{BS}$ at X, and $\overline{DQ}$ intersect $\overline{CS}$ at Y. Prove that lines $\overline{PR}$ and $\overline{XY}$ are either parallel or coincide.

Video

https://youtu.be/dC9VpiGVRqs

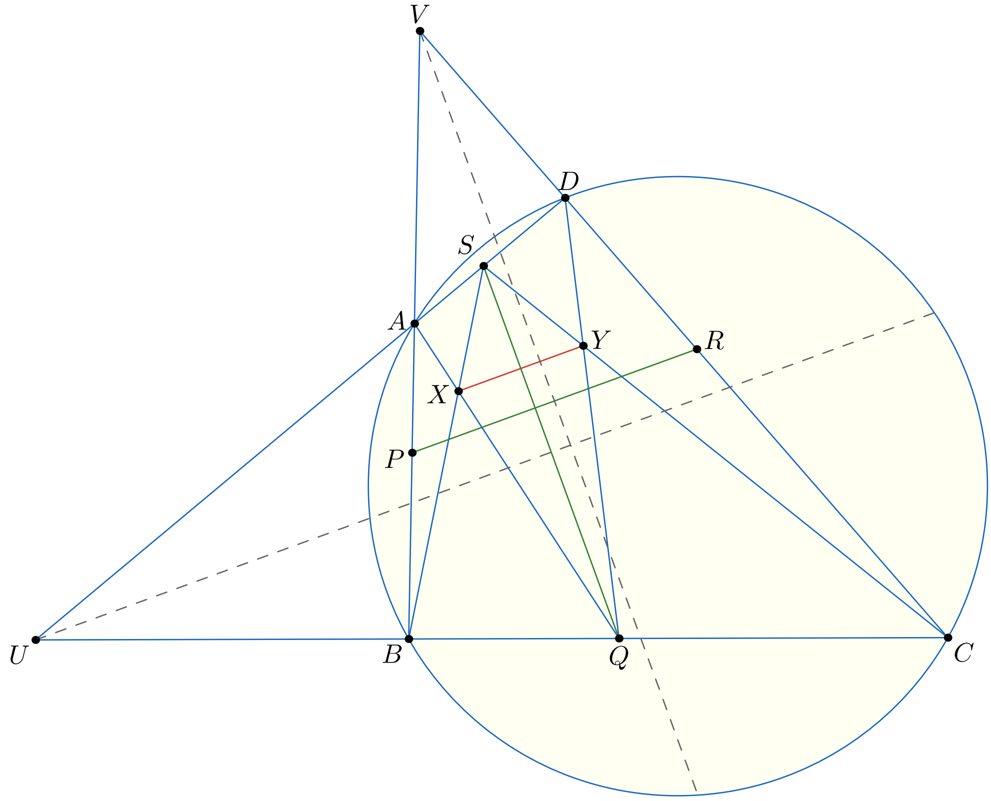
External Link

https://aops.com/community/p26379724

Solution

We present two approaches. The first is based on the points $U = \overline{AD} \cap \overline{BC}$ and $V = \overline{AB} \cap \overline{CD}$. The latter is based on $E = \overline{AC} \cap \overline{BD}$.

First solution (author's) Let $U = \overline{AD} \cap \overline{BC}$ and $V = \overline{AB} \cap \overline{CD}$.



Claim. We have $US = UQ$ and $VP = VR$.

Proof. We have

$$\measuredangle BSA = \measuredangle BAS + \measuredangle SBA = \measuredangle BCD + \measuredangle DCS = \measuredangle BCS$$

hence

$$US^2 = UB \cdot UC.$$

Similarly, $UQ^2 = UA \cdot UD = UB \cdot UC$. So, $US = UQ$; similarly $VP = VR$. □

Claim. Quadrilateral $SXQY$ is a kite (with $SX = SY$ and $QX = QY$).

Proof. We have

$$\measuredangle BSQ = \measuredangle USQ - \measuredangle USB = \measuredangle SQU - \measuredangle SCB = \measuredangle QSC$$

so $\overline{SQ}$ bisects $\angle BSC$; similarly it bisects $\angle AQD$. □

Claim. The internal bisectors of $\angle U$ and $\angle V$ are perpendicular.

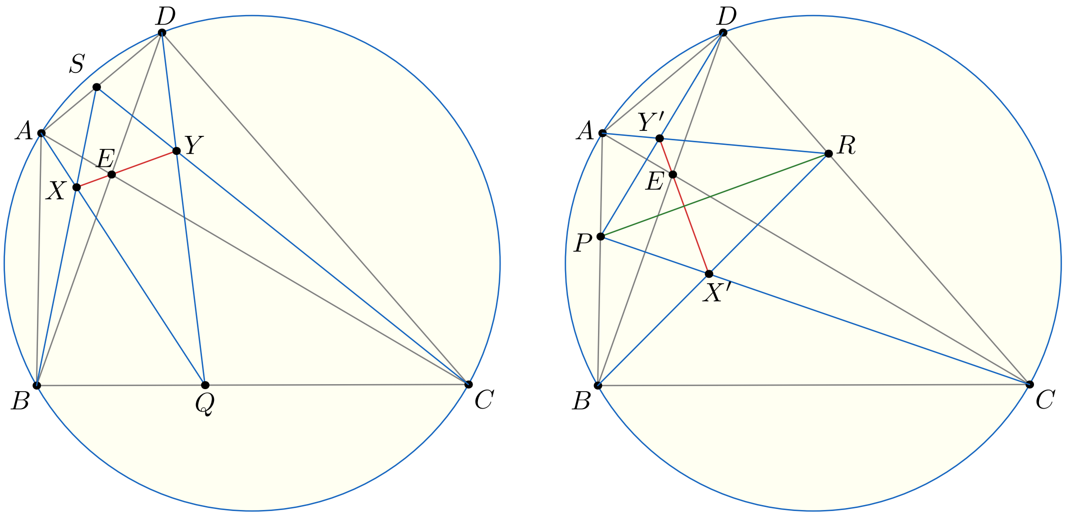
Proof. The angle between these angle bisectors equals

$$\begin{aligned} &\frac{1}{2}\angle DUC + \angle DAV + \frac{1}{2}\angle BVC \\ &= 90^\circ - \frac{\angle ADC}{2} - \frac{\angle DCB}{2} + \angle BCD + 90^\circ - \frac{\angle ABC}{2} - \frac{\angle DCB}{2} \\ &= 90^\circ. \end{aligned}$$

□

As $\overline{SQ}$ and $\overline{PR}$ are perpendicular to the internal bisectors of $\angle U$ and $\angle V$ by the first claim, so by the third claim $\overline{QS} \perp \overline{PR}$. Meanwhile the second claim gives that $\overline{XY}$ is perpendicular to $\overline{SQ}$, completing the problem.

Second solution due to Nikolai Beluhov Let $E = \overline{AC} \cap \overline{BD}$. Then E lies on $\overline{XY}$ by Pappus's theorem.



Claim. Line XEY is the interior bisector of $\angle AEB$ and $\angle CED$.

Proof. The angle conditions imply that X and Y are corresponding points in the two similar triangles AEB and DEC. Hence, $\angle AEX = \angle DEY$ and $\angle BEX = \angle CEY$. Since segments EX and EY are collinear, we're done. □

Introduce the points

$$X' = \overline{BR} \cap \overline{CP} \qquad \text{and} \qquad Y' = \overline{AR} \cap \overline{DP}.$$

By the same argument as before, line $X'EY'$ is the internal angle bisector of angles $\angle AED$ and $\angle BEC$.

Claim. Quadrilateral $PX'RY'$ is a kite (with $PX' = PY'$ and $RX' = RY'$).

Proof. Because X' and Y' are corresponding points in $\triangle BEC$ and $\triangle AED$,

$$\angle RX'Y' = 180^\circ - \angle BX'E = 180^\circ - \angle AY'E = \angle RY'X',$$

and so $RX' = RY'$. Similarly, $PX' = PY'$. □

Thus, $\overline{PR}$ is perpendicular to $\overline{X'EY'}$, hence parallel to the interior bisector of $\angle AEB$ and $\angle CED$. Together with the first claim, we're done.

Remark. It's possible to write up this solution without ever defining X' and Y'. The idea is to instead prove $SXQY$ is a kite (which is natural since X and Y are already marked) and hence obtain the sentence "$\overline{SQ}$ is parallel to the internal angle bisector of $\angle AED$ and $\angle BEC$" (using the first claim). Then cyclically shift the labels in to get the sentence "$\overline{PR}$ is parallel to the internal angle bisector of $\angle DEC$ and $\angle AEB$".