# USEMO 2022/4 

## Evan Chen

## Twitch Solves ISL

Episode 103

## Problem

Let $A B C D$ be a cyclic quadrilateral whose opposite sides are not parallel. Suppose points $P, Q, R, S$ lie in the interiors of segments $A B, B C, C D, D A$, respectively, such that

$$
\angle P D A=\angle P C B, \quad \angle Q A B=\angle Q D C, \quad \angle R B C=\angle R A D, \quad \text { and } \quad \angle S C D=\angle S B A .
$$

Let $\overline{A Q}$ intersect $\overline{B S}$ at $X$, and $\overline{D Q}$ intersect $\overline{C S}$ at $Y$. Prove that lines $\overline{P R}$ and $\overline{X Y}$ are either parallel or coincide.

## Video

https://youtu.be/dC9VpiGVRqs

## External Link

## Solution

We present two approaches. The first is based on the points $U=\overline{A D} \cap \overline{B C}$ and $V=\overline{A B} \cap \overline{C D}$. The latter is based on $E=\overline{A C} \cap \overline{B D}$.

First solution (author's). Let $U=\overline{A D} \cap \overline{B C}$ and $V=\overline{A B} \cap \overline{C D}$.


Claim. We have $U S=U Q$ and $V P=V R$.
Proof. We have

$$
\measuredangle B S A=\measuredangle B A S+\measuredangle S B A=\measuredangle B C D+\measuredangle D C S=\measuredangle B C S
$$

hence

$$
U S^{2}=U B \cdot U C .
$$

Similarly, $U Q^{2}=U A \cdot U D=U B \cdot U C$. So, $U S=U Q$; similarly $V P=V R$.
Claim. Quadrilateral $S X Q Y$ is a kite (with $S X=S Y$ and $Q X=Q Y$ ).
Proof. We have

$$
\measuredangle B S Q=\measuredangle U S Q-\measuredangle U S B=\measuredangle S Q U-\measuredangle S C B=\measuredangle Q S C
$$

so $\overline{S Q}$ bisects $\angle B S C$; similarly it bisects $\angle A Q D$.
Claim. The internal bisectors of $\angle U$ and $\angle V$ are perpendicular.
Proof. The angle between these angle bisectors equals

$$
\begin{aligned}
& \frac{1}{2} \angle D U C+\angle D A V+\frac{1}{2} \angle B V C \\
= & 90^{\circ}-\frac{\angle A D C}{2}-\frac{\angle D C B}{2}+\angle B C D+90^{\circ}-\frac{\angle A B C}{2}-\frac{\angle D C B}{2} \\
= & 90^{\circ} .
\end{aligned}
$$

As $\overline{S Q}$ and $\overline{P R}$ are perpendicular to the internal bisectors of $\angle U$ and $\angle V$ by the first claim, so by the third claim $\overline{Q S} \perp \overline{P R}$. Meanwhile the second claim gives that $\overline{X Y}$ is perpendicular to $\overline{S Q}$, completing the problem.

Second solution due to Nikolai Beluhov. Let $E=\overline{A C} \cap \overline{B D}$. Then $E$ lies on $\overline{X Y}$ by Pappus's theorem.


Claim. Line $X E Y$ is the interior bisector of $\angle A E B$ and $\angle C E D$.
Proof. The angle conditions imply that $X$ and $Y$ are corresponding points in the two similar triangles $A E B$ and $D E C$. Hence, $\angle A E X=\angle D E Y$ and $\angle B E X=\angle C E Y$. Since segments $E X$ and $E Y$ are collinear, we're done.

Introduce the points

$$
X^{\prime}=\overline{B R} \cap \overline{C P} \quad \text { and } \quad Y^{\prime}=\overline{A R} \cap \overline{D P} .
$$

By the same argument as before, line $X^{\prime} E Y^{\prime}$ is the internal angle bisector of angles $\angle A E D$ and $\angle B E C$.

Claim. Quadrilateral $P X^{\prime} R Y^{\prime}$ is a kite (with $P X^{\prime}=P Y^{\prime}$ and $R X^{\prime}=R Y^{\prime}$ ).
Proof. Because $X^{\prime}$ and $Y^{\prime}$ are corresponding points in $\triangle B E C$ and $\triangle A E D$,

$$
\angle R X^{\prime} Y^{\prime}=180^{\circ}-\angle B X^{\prime} E=180^{\circ}-\angle A Y^{\prime} E=\angle R Y^{\prime} X^{\prime}
$$

and so $R X^{\prime}=R Y^{\prime}$. Similarly, $P X^{\prime}=P Y^{\prime}$.
Thus, $\overline{P R}$ is perpendicular to $\overline{X^{\prime} E Y^{\prime}}$, hence parallel to the interior bisector of $\angle A E B$ and $\angle C E D$. Together with the first claim, we're done.

Remark. It's possible to write up this solution without ever defining $X^{\prime}$ and $Y^{\prime}$. The idea is to instead prove $S X Q Y$ is a kite (which is natural since $X$ and $Y$ are already marked) and hence obtain the sentence " $\overline{S Q}$ is parallel to the internal angle bisector of $\angle A E D$ and $\angle B E C$ " (using the first claim). Then cyclically shift the labels in to get the sentence " $\overline{P R}$ is parallel to the internal angle bisector of $\angle D E C$ and $\angle A E B$ ".

