IMO 1998/4

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Twitch Solves ISL

Episode 102

Problem

Determine all pairs (x, y) of positive integers such that x^2y+x+y is divisible by xy^2+y+7.

Video

https://youtu.be/Gv7-k9dXa6s

Solution

The answer is $(7k^2, 7k)$ for all $k \geq 1$, as well as $(11, 1)$ and $(49, 1)$.

We are given $xy^2 + y + 7 \mid x^2y + x + y$. Multiplying the right-hand side by y gives

$$xy^2 + y + 7 \mid x^2y^2 + xy + y^2$$

Then subtracting x times the left-hand side gives

$$xy^2 + y + 7 \mid y^2 - 7x.$$

We consider cases based on the sign of $y^2 = 7x$.

- If $y^2 > 7x$, then $0 < y^2 - 7x < xy^2 + y + 7$, contradiction.
- If $y^2 = 7x$, let $y = 7k$, so $x = 7k^2$. Plugging this back in to the original equation reads
$$343k^4 + 7k + 7 \mid 343k^5 + 7k^2 + 7k$$
which is always valid, hence these are all solutions.
- If $y^2 < 7x$, then $|y^2 - 7x| \leq 7x$, so $y \in \{1, 2\}$.

  When $y = 1$ we get
$$x + 8 \mid x^2 + x + 1 \iff x + 8 \mid 64 - 8 + 1 = 57.$$
  This has solutions $x = 11$ and $x = 49$.

  When $y = 2$
$$\begin{aligned} 4x + 9 &\mid 2x^2 + x + 2 \\ &\implies 4x + 9 \mid 16x^2 + 8x + 16 \\ &\implies 4x + 9 \mid 81 - 18 + 16 = 79 \end{aligned}$$
  which never occurs.