# Poland 2019/1/12 <br> Evan Chen 

## Twitch Solves ISL

Episode 100

## Problem

Let all numbers of form $x^{2}+y^{2}$ where $x, y$ are coprime integers be arranged in a sequence $z_{1}<z_{2}<z_{3}<\cdots$. (So the sequence begins $z_{1}=2=1^{1}+1^{2}, z_{2}=5=1^{2}+2^{2}$, $z_{3}=10=1^{2}+3^{2}, z_{4}=13=2^{2}+3^{2}$.) Prove that there exist infinitely many values of $n$ such that $z_{n}, z_{n+1}, \ldots, z_{n+2019}$ are odd.

## Video

https://youtu.be/8edyF716Mr8

## External Link

https://aops.com/community/p24529614

## Solution

Call numbers in the sequence good, and the other positive integers bad. We will need the following number theoretic facts.

Claim. Any number with a $3 \bmod 4$ factor is bad.
Proof. Follows directly from Fermat's Christmas theorem or quadratic reciprocity.
Claim. Let $t$ be a positive integer which is not a perfect square. Then there exists infinitely many primes $p$ such that $p \equiv 3(\bmod 4)$ and $-t$ is a quadratic residue modulo $p$.

Proof. By using quadratic reciprocity to show that some residue class works out, then using Dirichlet's theorem.

Pick 2020 odd primes $q_{0}<q_{1}<\cdots<q_{2019}$. We will find a positive integer $A$ such that

- $A^{2}+q_{0}^{2}, A^{2}+q_{1}^{2}, \ldots, A^{2}+q_{2019}^{2}$ are all good.
- $A^{2}+2, A^{2}+4, A^{2}+6, \ldots, A^{2}+\left(q_{2019}^{2}-1\right)$ are all bad.

This takes three steps:

1. We require $A \perp q_{0} q_{1} \ldots q_{2019}$. This guarantees the goodness of $A^{2}+q_{i}^{2}$.
2. We require $2 \mid A$. This means $A^{2}+s^{2}$ is bad for any even integer $s$.
3. For each even non-square $t$ in $\left[2, q_{2019}^{2}-1\right]$, we pick a prime $p_{t}$, different from any previously chosen prime, such that $-t$ is a quadratic residue modulo $p_{t}$ and $p_{t} \equiv 3$ $(\bmod 4)$. Then we require $A+t \equiv 0\left(\bmod p_{t}\right)$; this guarantees $A+t$ is bad.
