Poland 2019/1/12

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Twitch Solves ISL

Episode 100

Problem

Let all numbers of form $x^2 + y^2$ where x, y are coprime integers be arranged in a sequence $z_1 < z_2 < z_3 < \dots$. (So the sequence begins $z_1 = 2 = 1^1 + 1^2$, $z_2 = 5 = 1^2 + 2^2$, $z_3 = 10 = 1^2 + 3^2$, $z_4 = 13 = 2^2 + 3^2$.) Prove that there exist infinitely many values of n such that z_n, z_{n+1}, $\dots$, z_{n+2019} are odd.

Video

https://youtu.be/8edyF716Mr8

Solution

Call numbers in the sequence *good*, and the other positive integers *bad*. We will need the following number theoretic facts.

Claim. Any number with a 3 mod 4 factor is bad.

Proof. Follows directly from Fermat's Christmas theorem or quadratic reciprocity. □

Claim. Let t be a positive integer which is not a perfect square. Then there exists infinitely many primes p such that $p \equiv 3 \pmod 4$ and $-t$ is a quadratic residue modulo p.

Proof. By using quadratic reciprocity to show that some residue class works out, then using Dirichlet's theorem. □

Pick 2020 odd primes $q_0 < q_1 < \cdots < q_{2019}$. We will find a positive integer A such that

- $A^2 + q_0^2$, $A^2 + q_1^2$, ..., $A^2 + q_{2019}^2$ are all good.
- $A^2 + 2$, $A^2 + 4$, $A^2 + 6$, ..., $A^2 + (q_{2019}^2 - 1)$ are all bad.

This takes three steps:

1. We require $A \perp q_0 q_1 \dots q_{2019}$. This guarantees the goodness of $A^2 + q_i^2$.
2. We require $2 \mid A$. This means $A^2 + s^2$ is bad for any even integer s.
3. For each even non-square t in $[2, q_{2019}^2 - 1]$, we pick a prime p_t, different from any previously chosen prime, such that $-t$ is a quadratic residue modulo p_t and $p_t \equiv 3 \pmod 4$. Then we require $A + t \equiv 0 \pmod{p_t}$; this guarantees $A + t$ is bad.