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Twitch Solves ISL

Episode 99

Problem

We say that two sequences of positive integers $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are completely different if $x_n \neq y_n$ for all $n \in \mathbb{N}$. Let F be a function assigning a positive integer to every sequence of positive integers such that $F(x) \neq F(y)$ for any pair of completely different sequences x, y, and for constant sequences we have $F((k, k, \dots)) = k$. Prove that there exists n such that $F(x) = x_n$ for all sequences x.

Video

https://youtu.be/i1d3h_Rs9Iw

Solution

By permuting indices suitably, we are going to assume $F(1, 2, 3, \dots) = 1$ and show that F returns the first value.

Claim. $F(x_\bullet)$ always returns an element that appears in $x_\bullet$.

Proof. If $F(x_\bullet) = k$ and $k \notin x_\bullet$, then $x_\bullet$ is completely different from $(k, k, \dots)$, yet $F(k, k, \dots) = k$. □

Claim. If $x_\bullet$ is a sequence that contains only two different numbers, then $F(x_\bullet) = x_1$.

Proof. Call the two numbers $x_1 = a$ and b. First, we resolve the case where $a, b \geq 2$. Indeed, note that

$$F(b, 1, 1, 1, \dots) = b$$

since that sequence is completely different from $(1, 2, 3, \dots)$ so its F-value cannot be 1, hence must be b. But $(b, 1, 1, \dots)$ is completely different from $(x_\bullet)$ so we get $F(x_\bullet) = a$ as desired.

To deal with the case $a = 1$, we pick any third number c not equal to 1 or b, and consider the sequence

$$y_n = \begin{cases} b & x_n = 1 \\ c & x_n = b. \end{cases}$$

Now $F(y_n) = b$, and y_n is completely different from x_n. So this forces $F(x_n) = a$.

Similarly, to deal with the case $b = 1$, we pick any third number c not equal to a or 1, and consider the sequence

$$y_n = \begin{cases} b & x_n = a \\ c & x_n = 1. \end{cases}$$

Now $F(y_n) = b$, and y_n is completely different from x_n. So this forces $F(x_n) = a$ too. □

We now finish the general case. Suppose for contradiction that

$$F(a_1, a_2, \dots) = r \neq a_1.$$

Then we can construct a sequence b_n by

$$b_n = \begin{cases} r & a_n = a_1 \\ a_1 & \text{otherwise} \end{cases}$$

By construction, a_n is completely different from b_n. But the previous claim gives $F(b_n) = r$, contradiction.