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Twitch Solves ISL

Episode 99

Problem

For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4; $2+2$; $2+1+1$; $1+1+1+1$.

Prove that for any integer $n \ge 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.

Solution

It's clear that f is non-decreasing. By sorting by the number of 1's we used, we have the equation

$$f(N)=f\left(\left\lfloor\frac{N}{2}\right\rfloor\right)+f\left(\left\lfloor\frac{N}{2}\right\rfloor-1\right)+f\left(\left\lfloor\frac{N}{2}\right\rfloor-2\right)+\cdots+f(1)+f(0). \quad (\bigstar)$$

Upper bound. We now prove the upper bound by induction. Indeed, the base case is trivial and for the inductive step we simply use $(\bigstar)$:

$$f(2^n)=f(2^{n-1})+f(2^{n-1}-1)+\cdots<2^{n-1}f(2^{n-1})<2^{n-1}\cdot 2^{\frac{(n-1)^2}{2}}=2^{\frac{n^2}{2}-\frac{1}{2}}.$$

Lower bound. First, we contend that f is convex. We'll first prove this in the even case to save ourselves some annoyance:

Claim (f is basically convex)**.** If $2 \mid a+b$ then we have $f(2a)+f(2b)\ge 2f\,(a+b)$.

Proof. Since $f(2k+1)=f(2k)$, we will only prove the first equation. Assume WLOG $a\ge b$ and use $(\bigstar)$ on all three f expressions here; after subtracting repeated terms, the inequality then rewrites as

$$\sum_{(a+b)/2\le x\le a} f(x) \ge \sum_{b\le x\le (a+b)/2} f(x).$$

This is true since there are an equal number of terms on each side and f is nondecreasing. □

Claim. For each $1\le k<2^{n-1}$, we have

$$f(2^{n-1}-k)+f(k+1)\ge 2f(2^{n-2})$$

Proof. Use the fact that $f(2t+1)=f(2t)$ for all t and then apply convexity as above. □

Now we can carry out the induction:

$$f(2^n)=f(2^{n-1})+f(2^{n-1}-1)+\cdots>2^{n-1}f(2^{n-2})+f(0)>2^{n-1}2^{\frac{(n-2)^2}{4}}=2^{\frac{n^2}{4}}.$$