

IMO 1997/6

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TWITCH SOLVES ISL

Episode 99

Problem

For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways: 4 ; $2 + 2$; $2 + 1 + 1$; $1 + 1 + 1 + 1$.

Prove that for any integer $n \geq 3$ we have $2^{\frac{n^2}{4}} < f(2^n) < 2^{\frac{n^2}{2}}$.

External Link

<https://aops.com/community/p356713>

Solution

It's clear that f is non-decreasing. By sorting by the number of 1's we used, we have the equation

$$f(N) = f\left(\left\lfloor \frac{N}{2} \right\rfloor\right) + f\left(\left\lfloor \frac{N}{2} \right\rfloor - 1\right) + f\left(\left\lfloor \frac{N}{2} \right\rfloor - 2\right) + \cdots + f(1) + f(0). \quad (\star)$$

Upper bound. We now prove the upper bound by induction. Indeed, the base case is trivial and for the inductive step we simply use (\star) :

$$f(2^n) = f(2^{n-1}) + f(2^{n-1} - 1) + \cdots < 2^{n-1} f(2^{n-1}) < 2^{n-1} \cdot 2^{\frac{(n-1)^2}{2}} = 2^{\frac{n^2}{2} - \frac{1}{2}}.$$

Lower bound. First, we contend that f is convex. We'll first prove this in the even case to save ourselves some annoyance:

Claim (f is basically convex). If $2 \mid a + b$ then we have $f(2a) + f(2b) \geq 2f(a + b)$.

Proof. Since $f(2k + 1) = f(2k)$, we will only prove the first equation. Assume WLOG $a \geq b$ and use (\star) on all three f expressions here; after subtracting repeated terms, the inequality then rewrites as

$$\sum_{(a+b)/2 \leq x \leq a} f(x) \geq \sum_{b \leq x \leq (a+b)/2} f(x).$$

This is true since there are an equal number of terms on each side and f is nondecreasing. \square

Claim. For each $1 \leq k < 2^{n-1}$, we have

$$f(2^{n-1} - k) + f(k + 1) \geq 2f(2^{n-2})$$

Proof. Use the fact that $f(2t + 1) = f(2t)$ for all t and then apply convexity as above. \square

Now we can carry out the induction:

$$f(2^n) = f(2^{n-1}) + f(2^{n-1} - 1) + \cdots > 2^{n-1} f(2^{n-2}) + f(0) > 2^{n-1} 2^{\frac{(n-2)^2}{4}} = 2^{\frac{n^2}{4}}.$$