IMO 1997/4

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Twitch Solves ISL

Episode 99

Problem

An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \dots, n$, the i-th row and the i-th column together contain all elements of S. Show that:

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

Video

https://youtu.be/qMGDgxJqzIU

Solution

For (a), define a *cross* to be the union of the ith row and ith column. Every cell of the matrix not on the diagonal is contained in exactly two crosses, while each cell on the diagonal is contained in one cross.

On the other hand, if a silver matrix existed for $n = 1997$, then each element of S is in all 1997 crosses, so it must appear at least once on the diagonal since 1997 is odd. However, $|S| = 3993$ while there are only 1997 diagonal cells. This is a contradiction.

For (b), we construct a silver matrix M_e for $n = 2^e$ for each $e \geq 1$. We write the first three explicitly for concreteness:

$$M_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 3 & 1 & 6 & 7 \\ 7 & 5 & 1 & 2 \\ 6 & 4 & 3 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 2 & 4 & 5 & 8 & 9 & 11 & 12 \\ 3 & 1 & 6 & 7 & 10 & 15 & 13 & 14 \\ 7 & 5 & 1 & 2 & 14 & 12 & 8 & 9 \\ 6 & 4 & 3 & 1 & 13 & 11 & 10 & 15 \\ 15 & 9 & 11 & 12 & 1 & 2 & 4 & 5 \\ 10 & 8 & 13 & 14 & 3 & 1 & 6 & 7 \\ 14 & 12 & 15 & 9 & 7 & 5 & 1 & 2 \\ 13 & 11 & 10 & 8 & 6 & 4 & 3 & 1 \end{bmatrix}$$

The construction is described recursively as follows. Let

$$M_e' = \left[\begin{array}{c|c} M_{e-1} & M_{e-1} + (2^e - 1) \\ \hline M_{e-1} + (2^e - 1) & M_{e-1} \end{array} \right].$$

Then to get from M_e' to M_e, replace half of the 2^e's with $2^{e+1} - 1$: in the northeast quadrant, the even-indexed ones, and in the southwest quadrant, the odd-indexed ones.