# IMO 1997/4 <br> Evan Chen 

Twitch Solves ISL
Episode 99

## Problem

An $n \times n$ matrix whose entries come from the set $S=\{1,2, \ldots, 2 n-1\}$ is called a silver matrix if, for each $i=1,2, \ldots, n$, the $i$-th row and the $i$-th column together contain all elements of $S$. Show that:
(a) there is no silver matrix for $n=1997$;
(b) silver matrices exist for infinitely many values of $n$.

## Video

https://youtu.be/qMGDgxJqzIU

## External Link

https://aops.com/community/p611

## Solution

For (a), define a cross to be the union of the $i$ th row and $i$ th column. Every cell of the matrix not on the diagonal is contained in exactly two crosses, while each cell on the diagonal is contained in one cross.

On the other hand, if a silver matrix existed for $n=1997$, then each element of $S$ is in all 1997 crosses, so it must appear at least once on the diagonal since 1997 is odd. However, $|S|=3993$ while there are only 1997 diagonal cells. This is a contradiction.

For (b), we construct a silver matrix $M_{e}$ for $n=2^{e}$ for each $e \geq 1$. We write the first three explicitly for concreteness:

$$
\begin{aligned}
M_{1} & =\left[\begin{array}{ll}
1 & 2 \\
3 & 1
\end{array}\right] \\
M_{2} & =\left[\begin{array}{llll}
1 & 2 & 4 & 5 \\
3 & 1 & 6 & 7 \\
7 & 5 & 1 & 2 \\
6 & 4 & 3 & 1
\end{array}\right] \\
M_{3} & =\left[\begin{array}{ccccccc}
1 & 2 & 4 & 5 & 8 & 9 & 11 \\
3 & 1 & 6 & 7 & 10 & 15 & 13 \\
7 & 5 & 1 & 2 & 14 & 12 & 8 \\
\hline 6 & 4 & 3 & 1 & 13 & 11 & 10 \\
15 & 9 & 11 & 12 & 1 & 2 & 4 \\
5 \\
10 & 8 & 13 & 14 & 3 & 1 & 6 \\
14 & 12 & 15 & 9 & 7 & 5 & 1 \\
13 & 11 & 10 & 8 & 6 & 4 & 3 \\
1
\end{array}\right]
\end{aligned}
$$

The construction is described recursively as follows. Let

$$
M_{e}^{\prime}=\left[\begin{array}{c|c}
M_{e-1} & M_{e-1}+\left(2^{e}-1\right) \\
\hline M_{e-1}+\left(2^{e}-1\right) & M_{e-1}
\end{array}\right] .
$$

Then to get from $M_{e}^{\prime}$ to $M_{e}$, replace half of the $2^{e}$ 's with $2^{e+1}-1$ : in the northeast quadrant, the even-indexed ones, and in the southwest quadrant, the odd-indexed ones.

Remark. In fact, it turns out silver matrices exist for all even dimensions. A claimed proof is outlined at https://aops.com/community/p7375020.

