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Twitch Solves ISL

Episode 98

Problem

A set S of points from the space will be called completely symmetric if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

Video

https://youtu.be/QIkoSVCHrR8

Solution

Let G be the centroid of S.

Claim. All points of S lie on a sphere Γ centered at G.

Proof. Each perpendicular bisector plane passes through G. So if $A, B \in S$ it follows $GA = GB$. □

Claim. Consider any plane passing through three or more points of S. The points of S in the plane form a regular polygon.

Proof. The cross section is a circle because we are intersecting a plane with sphere Γ. Now if A, B, C are three adjacent points on this circle, by taking the perpendicular bisector we have $AB = BC$. □

If the points of S all lie in a plane, we are done. Otherwise, the points of S determine a polyhedron Π inscribed in Γ. All of the faces of Π are evidently regular polygons, of the same side length s.

Claim. Every face of Π is an equilateral triangle.

Proof. Suppose on the contrary some face $A_1A_2 \dots A_n$ has $n > 3$. Let B be any vertex adjacent to A_1 in Π other than A_2 or A_n. Consider the plane determined by $\triangle A_1A_3B$. This is supposed to be a regular polygon, but arc A_1A_3 is longer than arc A_1B, and by construction there are no points inside these arcs. This is a contradiction. □

Hence, Π has faces all congruent equilateral triangles. This implies it is a regular polyhedron — either a regular tetrahedron, regular octahedron, or regular icosahedron. We can check the regular icosahedron fails by taking two antipodal points as our counterexample. This finishes the problem.