gcd det

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Twitch Solves ISL

Episode 97

Problem

Let n be a given positive integer. Find the determinant of the $n \times n$ matrix whose (i, j)th entry is $\gcd(i, j)$.

Video

https://youtu.be/SnEj4CljU18

Solution

The answer is the product $\varphi(1)\varphi(2)\dots\varphi(n)$, where φ is the Euler phi function.

The proof is by induction on n, with base cases immediate. The main claim is that we can eliminate nearly all of the entries in the last column by doing column operations. To be precise:

Claim. Let $n = p_1^{e_1} \dots p_k^{e_k}$. Then for any integer $1 \le m \le n$, we have

$$\sum_{S \subseteq \{1,2,\dots,k\}} (-1)^{|S|} \gcd\left(m, \frac{n}{\prod_{i \in S} p_i}\right) = \begin{cases} 0 & m < n \\ \varphi(n) & m = n. \end{cases}$$

Hence, if the claim is true, then we can take the $\frac{n}{\prod_{i \in S} p_i}$'th columns for each nonempty S and add/subtract them from the nth column (subtraction if $|S|$ is odd, addition if $|S|$ is even). This would leave all zeros except $\varphi(n)$ in the bottom right. Since we didn't touch any of the other rows or columns, the original problem is solved by induction.

Proof of claim. Let $m = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$. The sum can be factored in the following way:

$$\prod_{i=1}^{k} \left(p_i^{\min(a_i, e_i)} - p_i^{\min(a_i, e_i - 1)}\right).$$

If $m = n$, then $a_i = e_i$ and the above product becomes the usual factorization of the Euler phi function.

On the other hand, if $m < n$, then there exists some particular index i such that $a_i < e_i$. That makes the corresponding term in the product zero, and so the whole product vanishes. □