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## Problem

Peter and Paul play the following game. First, Peter chooses some positive integer $a$ with the sum of its digits equal to 2012. Paul wants to determine this number; he knows only that the sum of the digits of Peter's number is 2012. On each of his moves Paul chooses a positive integer $x$ and Peter tells him the sum of the digits of $|x-a|$. What is the minimal number of moves in which Paul can determine Peter's number for sure?

## Video

https://youtu.be/K759pczb7-A

## External Link

https://aops.com/community/p14404801

## Solution

The answer is 2012, and in general, if 2012 is replaced by $n$ the answer is $n$.
Let $N$ denote Peter's number. For Paul's algorithm, Paul can start by picking 1. Then Peter's reply will be exactly $n-1+9 k$, where $k$ is the number of trailing zeros at the end of $N$. So, Paul learns the position of the rightmost nonzero digit of $N$. Now, observe that if Paul increases all his future guesses by $10^{k}$, he is essentially playing the same game with $N-10^{k}$. This is a number with sum of digits $n-1$, so induction gets the desired result.

Now we show $n$ moves are needed. Suppose Peter pre-commits to choosing a number of the form

$$
a=1 \underbrace{0 \ldots 0}_{x_{1}} 1 \underbrace{0 \ldots 0}_{x_{2}} 1 \underbrace{0 \ldots 0}_{x_{3}} \ldots 1 \underbrace{0 \ldots 0}_{x_{n}}
$$

and even announces this to Paul. On Paul's turn, he must start by picking some number, and if his number turns is less than $10^{x_{n}}$, he will only learn the value of $x_{n}$. The same argument applies with $10^{x_{n-1}+1+x_{n}}$ on the next turn, in which Paul's second turn only tells him the value of $x_{n-1}$ if his number is not large enough. And so on. This shows that at least $n$ moves are necessary.

