ToT Fall 2012 J-A7

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Twitch Solves ISL

Episode 97

Problem

Peter and Paul play the following game. First, Peter chooses some positive integer a with the sum of its digits equal to 2012. Paul wants to determine this number; he knows only that the sum of the digits of Peter's number is 2012. On each of his moves Paul chooses a positive integer x and Peter tells him the sum of the digits of $|x - a|$. What is the minimal number of moves in which Paul can determine Peter's number for sure?

Video

https://youtu.be/K759pczb7-A

External Link

https://aops.com/community/p14404801

Solution

The answer is 2012, and in general, if 2012 is replaced by n the answer is n.

Let N denote Peter's number. For Paul's algorithm, Paul can start by picking 1. Then Peter's reply will be exactly $n - 1 + 9k$, where k is the number of trailing zeros at the end of N. So, Paul learns the position of the rightmost nonzero digit of N. Now, observe that if Paul increases all his future guesses by 10^k, he is essentially playing the same game with $N - 10^k$. This is a number with sum of digits $n - 1$, so induction gets the desired result.

Now we show n moves are needed. Suppose Peter pre-commits to choosing a number of the form

$$a = 1\underbrace{0\dots0}_{x_1}1\underbrace{0\dots0}_{x_2}1\underbrace{0\dots0}_{x_3}\dots1\underbrace{0\dots0}_{x_n}$$

and even announces this to Paul. On Paul's turn, he must start by picking *some* number, and if his number turns is less than 10^{x_n}, he will only learn the value of x_n. The same argument applies with $10^{x_{n-1}+1+x_n}$ on the next turn, in which Paul's second turn only tells him the value of x_{n-1} if his number is not large enough. And so on. This shows that at least n moves are necessary.