

Sylver coinage

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TWITCH SOLVES ISL

Episode 95

Problem

Two players alternate naming positive integers. The first player to name 1 or a sum of some previous numbers (possibly with repetition) loses. Prove that the first player wins.

Solution

Call the players Alice and Bob, by tradition. Let Alice start with some prime $p \geq 5$, and let Bob's reply be the integer q which is relatively prime (but not necessarily itself a prime).

We now employ strategy stealing on the third turn. Alice considers what happens if she plays $pq - p - q$. If she still loses to a sequence of moves from Bob, then she uses that sequence, instead of $pq - p - q$. So in order to complete the proof we need to show:

Claim 1. After the first stolen move x from Bob, $pq - p - q$ is invalidated anyways by the combination of p , q , and the move x .

Proof. Since x is required to not be a linear combination of p and q , we have $x < pq - p - q$. We claim that $(pq - p - q) - x$, which is nonnegative, is in fact a sum of p 's and q 's. That's because the Chicken McNugget theorem actually says more strongly that for any $x \leq pq - p - q$, exactly one of x and $(pq - p - q) - x$ is a sum of p 's and q 's. \square