

Sylver coinage

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TWITCH SOLVES ISL

Episode 95

Problem

Two players alternate naming positive integers. The first player to name 1 or a sum of some previous numbers (possibly with repetition) loses. Determine which player has the winning strategy.

Solution

Call the players Alice and Bob, by tradition. We claim Alice wins.

Let Alice start with some prime $p \geq 5$, and let Bob's reply be the integer q which is relatively prime (but not necessarily itself a prime).

We now employ strategy stealing on the third turn. Alice considers what happens if she plays $pq - p - q$. If she still loses to a sequence of moves from Bob, then she uses that sequence, instead of $pq - p - q$. So in order to complete the proof we need to show:

Claim. After the first stolen move x from Bob, $pq - p - q$ is invalidated anyways by the combination of p , q , and the move x .

Proof. Since x is required to not be a linear combination of p and q , we have $x < pq - p - q$. We claim that $(pq - p - q) - x$, which is nonnegative, is in fact a sum of p 's and q 's. That's because the Chicken McNugget theorem actually says more strongly that for any $x \leq pq - p - q$, exactly one of x and $(pq - p - q) - x$ is a sum of p 's and q 's. \square