Twitch 091.2

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Twitch Solves ISL

Episode 91

Problem

Suppose there are N blocks with dimensions $1 \times 2 \times 4$ which are placed in an axis-aligned way in a $(2n+1) \times (2n+1) \times (2n+1)$ cube. (The blocks may be rotated.) Prove that if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$ then

$$N \le \frac{n(n+1)(2n+1)}{2} - 1.$$

Video

https://youtu.be/xrAkxr6mB9w

Solution

Call a $1 \times 2 \times 4$ block an *ice tray*. First, note that in every $(2n+1) \times (2n+1) \times 1$ cross section, any ice tray intersects the cross section in an even number of cells. This immediately implies there are $2n+1$ cells not in an ice tray, giving a bound of

$$N \le \frac{(2n+1)^3 - (2n+1)}{8}$$

which is one more than the requested bound. So we have to show this is not always sharp.

Impose coordinates $\{0, 1, \dots, 2n\}^3$. Assume for contradiction there is an equality case above and let S be the coordinates of those $2n+1$ *missing* cells. We also will say a missing cell is *even* or *odd* according to the parity of the sum of its coordinates.

Then consider the generating function

$$F(X) = (X^0 + X^1 + \dots + X^{2n})^3 - \sum_{(a,b,c) \in S} X^{a+b+c}.$$

This is the sum of $X^{\text{sum coords}}$ over all cells. Because the cube was partitioned into ice trays, it follows that that F is divisible by $(1+X)(1+X+X^2+X^3)$, as this polynomial divides the contribution of any ice tray.

In particular, if $n \equiv 1 \pmod 4$ then

$$0 = F(-1) = 1 - \sum_{(a,b,c)} (-1)^{a+b+c}$$
$$0 = F(i) = 1 - \sum_{(a,b,c)} i^{a+b+c}.$$

The first equation implies there are $n+1$ even missing cells, and n odd missing cells. But then the second equation is not possible, because the real component of the sum should have the same parity as $n+1$.

Similarly, if $n \equiv 2 \pmod 4$ then

$$0 = F(-1) = 1 - \sum_{(a,b,c)} (-1)^{a+b+c}$$
$$0 = F(i) = i - \sum_{(a,b,c)} i^{a+b+c}.$$

The first equation implies there are $n+1$ even missing cells, and n odd missing cells. But then the second equation is not possible, because the real component of the sum should have the same parity as $n+1$.