# Twitch 091.2 

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Twitch Solves ISL
Episode 91

## Problem

Suppose there are $N$ blocks with dimensions $1 \times 2 \times 4$ which are placed in an axis-aligned way in a $(2 n+1) \times(2 n+1) \times(2 n+1)$ cube. (The blocks may be rotated.) Prove that if $n \equiv 1(\bmod 4)$ or $n \equiv 2(\bmod 4)$ then

$$
N \leq \frac{n(n+1)(2 n+1)}{2}-1
$$

## Video

https://youtu.be/xrAkxr6mB9w

## Solution

Call a $1 \times 2 \times 4$ block an ice tray. First, note that in every $(2 n+1) \times(2 n+1) \times 1$ cross section, any ice tray intersects the cross section in an even number of cells. This immediately implies there are $2 n+1$ cells not in an ice tray, giving a bound of

$$
N \leq \frac{(2 n+1)^{3}-(2 n+1)}{8}
$$

which is one more than the requested bound. So we have to show this is not always sharp.

Impose coordinates $\{0,1, \ldots, 2 n\}^{3}$. Assume for contradiction there is an equality case above and let $S$ be the coordinates of those $2 n+1$ missing cells. We also will say a missing cell is even or odd according to the parity of the sum of its coordinates.

Then consider the generating function

$$
F(X)=\left(X^{0}+X^{1}+\cdots+X^{2 n}\right)^{3}-\sum_{(a, b, c) \in S} X^{a+b+c} .
$$

This is the sum of $X^{\text {sum coords }}$ over all cells. Because the cube was partitioned into ice trays, it follows that that $F$ is divisible by $(1+X)\left(1+X+X^{2}+X^{3}\right)$, as this polynomial divides the contribution of any ice tray.

In particular, if $n \equiv 1(\bmod 4)$ then

$$
\begin{gathered}
0=F(-1)=1-\sum_{(a, b, c)}(-1)^{a+b+c} \\
0=F(i)=1-\sum_{(a, b, c)} i^{a+b+c} .
\end{gathered}
$$

The first equation implies there are $n+1$ even missing cells, and $n$ odd missing cells. But then the second equation is not possible, because the real component of the sum should have the same parity as $n+1$.

Similarly, if $n \equiv 2(\bmod 4)$ then

$$
\begin{gathered}
0=F(-1)=1-\sum_{(a, b, c)}(-1)^{a+b+c} \\
0=F(i)=i-\sum_{(a, b, c)} i^{a+b+c} .
\end{gathered}
$$

The first equation implies there are $n+1$ even missing cells, and $n$ odd missing cells. But then the second equation is not possible, because the real component of the sum should have the same parity as $n+1$.

