MR J560

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Twitch Solves ISL

Episode 91

Problem

Show that for all positive reals a, b, c,

$$\frac{2}{a^2} + \frac{5}{b^2} + \frac{45}{c^2} > \frac{16}{(a+b)^2} + \frac{24}{(b+c)^2} + \frac{48}{(c+a)^2}.$$

Video

https://youtu.be/xrAkxr6mB9w

Solution

By Holder inequality we generally have

$$\frac{u}{x^2} + \frac{v}{y^2} \geq (u^{1/3} + v^{1/3})^3 \cdot \frac{1}{(x+y)^2}$$

for any u, v, x, y. Therefore, we will be done if we can find choices $0 \leq r \leq 2$, $0 \leq s \leq 5$, $0 \leq t \leq 45$ such that

$$\begin{aligned} r^{1/3} + (5-s)^{1/3} &\geq 2\sqrt[3]{2} = \sqrt[3]{16} \\ s^{1/3} + (45-t)^{1/3} &\geq 2\sqrt[3]{3} = \sqrt[3]{24} \\ t^{1/3} + (2-r)^{1/3} &\geq 2\sqrt[3]{6} = \sqrt[3]{48} \end{aligned}$$

with at least one inequality being strict.

The official solution uses the choices $(r, s, t) = (1, 1, 36)$ in which case each of the inequalities above is true by AM-GM.

Remark. More blunt approaches exist by simply randomly guessing, say:

$$\begin{aligned} r &= 2 \cdot 0.9^3 = 1.458 \\ s &= 5 - (1.1 \cdot \sqrt[3]{2})^3 = 5 - 2.662 = 2.338 \\ t &= 45 - (\sqrt[3]{24} - \sqrt[3]{2.338})^3 \\ &= 45 - 24 + 2.338 + 3\sqrt[3]{24^2 \cdot 2.338} - 3\sqrt[3]{24 \cdot 2.338^2} \\ &= 23.338 + 3\sqrt[3]{576 \cdot 2.338} - 3\sqrt[3]{24 \cdot 2.338^2} \\ &> 23.338 + 3\sqrt[3]{1346.688} - 3\sqrt[3]{138.24} \\ &> 23.338 + 33 - 16 = 39.338 \\ t^{1/3} + (2-r)^{1/3} &= \sqrt[3]{39.338} + \sqrt[3]{0.542}. \end{aligned}$$

So we wish to show

$$\sqrt[3]{39.338} + \sqrt[3]{.542} > \sqrt[3]{48}$$

but $\sqrt[3]{39.338} > 3.4$ since $3.4^3 = 11.56 \cdot 3.4 = 39.304$, and $\sqrt[3]{.542} > 0.8$, while plainly $\sqrt[3]{48} < 4$.