IMO 1999/6

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Twitch Solves ISL

Episode 91

Problem

Find all the functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Video

https://youtu.be/rlMW65i_VtI

Solution

The answer is $f(x) = -\frac{1}{2}x^2 + 1$ which obviously works.

For the other direction, first note that

$$P(f(y), y) \implies 2f(f(y)) + f(y)^2 - 1 = f(0).$$

We introduce the notation $c = \frac{f(0)-1}{2}$, and $S = \operatorname{img} f$. Then the above assertion says

$$f(s) = -\frac{1}{2}s^2 + (c+1).$$

Thus, the given functional equation can be rewritten as

$$Q(x,s) : f(x-s) = -\frac{1}{2}s^2 + sx + f(x) - c.$$

Claim (Main claim)**.** We can find a function $g \colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x-z) = zx + f(x) + g(z). \qquad (\spadesuit).$$

Proof. If $z \neq 0$, the idea is to fix a nonzero value $s_0 \in S$ (it exists) and then choose x_0 such that $-\frac{1}{2}s_0^2 + s_0x_0 - c = z$. Then, $Q(x_0, s)$ gives an pair (u, v) with $u - v = z$.

But now for any x, using $Q(x+v, u)$ and $Q(x, -v)$ gives

$$\begin{aligned} f(x-z) - f(x) = f(x-u+v) - f(x) &= f(x+v) - f(x) + u(x+v) - \frac{1}{2}u^2 + c \\ &= -vx - \frac{1}{2}s^2 - c + u(x+v) - \frac{1}{2}u^2 + c \\ &= -vx - \frac{1}{2}v^2 + u(x+v) - \frac{1}{2}u^2 = zx + g(z) \end{aligned}$$

where $g(z) = -\frac{1}{2}(u^2+v^2)$ depends only on z. $\square$

Now, let

$$h(x) \stackrel{\text{def}}{=} \frac{1}{2}x^2 + f(x) - (2c+1),$$

so $h(0) = 0$.

Claim. The function h is additive.

Proof. We just need to rewrite $(\spadesuit)$. Letting $x = z$ in $(\spadesuit)$, we find that actually $g(x) = f(0) - x^2 - f(x)$. Using the definition of h now gives

$$h(x-z) = h(x) + h(z). \qquad \square$$

To finish, we need to remember that f, hence h, is known on the image

$$S = \{f(x) \mid x \in \mathbb{R}\} = \left\{ h(x) - \frac{1}{2}x^2 + (2c+1) \mid x \in \mathbb{R} \right\}.$$

Thus, we derive

$$h\left(h(x) - \frac{1}{2}x^2 + (2c+1)\right) = -c \qquad \forall x \in \mathbb{R}. \qquad (\heartsuit)$$

We can take the following two instances of $\heartsuit$:

$$h\left(h(2x) - 2x^2 + (2c+1)\right) = -c$$
$$h\left(2h(x) - x^2 + 2(2c+1)\right) = -2c.$$

Now subtracting these and using $2h(x) = h(2x)$ gives

$$c = h\left(-x^2 - (2c+1)\right).$$

Together with h additive, this implies readily h is constant. That means $c = 0$ and the problem is solved.