USEMO 2021/6

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Twitch Solves ISL

Episode 90

Problem

A *bagel* is a loop of $2a+2b+4$ unit squares which can be obtained by cutting a concentric $a \times b$ hole out of an $(a+2) \times (b+2)$ rectangle, for some positive integers a and b. (The side of length a of the hole is parallel to the side of length $a+2$ of the rectangle.)

Consider an infinite grid of unit square cells. For each even integer $n \geq 8$, a *bakery of order* n is a finite set of cells S such that, for every n-cell bagel B in the grid, there exists a congruent copy of B all of whose cells are in S. (The copy can be translated and rotated.) We denote by $f(n)$ the smallest possible number of cells in a bakery of order n.

Find a real number α such that, for all sufficiently large even integers $n \geq 8$, we have

$$\frac{1}{100} < \frac{f(n)}{n^\alpha} < 100.$$

Video

https://youtu.be/PhNIee2CzdY

External Link

https://aops.com/community/p23524122

Solution

The answer is $\alpha = 3/2$.

In what follows, "Y is about X" means that $Y = [1+o(1)]X$. Equivalently, $\lim_{n\to\infty} Y/X = 1$. Intuitively, both of these say that X and Y become closer and closer together as n grows. This is fine for the problem since only sufficiently large n are involved.

Bound. First we prove that every bakery S of order n contains at least about $n^{3/2}/8$ cells.

We say that a bagel is *horizontal* or *vertical* depending on the orientation of its pair of longer sides. (A square bagel is both.) For each $a < b$ with $2a + 2b + 4 = n$, take one bagel in S whose hole is of size either $a \times b$ or $b \times a$. Without loss of generality, at least about $n/8$ of our bagels are horizontal.

Say that there are a total of k rows which contain a longer side of at least one of our horizontal bagels. Note that the shorter side length of a horizontal bagel depends only on the distance between the rows of its longer sides. Since the shorter side lengths of all of our bagels are pairwise distinct, we obtain that $\binom{k}{2}$ is at least about $n/8$. Consequently, k is at least about $\sqrt{n}/2$.

On the other hand, each such row contains at least about $n/4$ cells in S. Therefore, $|S|$ is at least about $n^{3/2}/8$, as needed.

Construction. To complete the solution, we construct a bakery S of order n with at most about $\sqrt{2} \cdot n^{3/2}$ cells. Define

$$\ell = \left\lceil \sqrt{n/2} \right\rceil \qquad \text{and} \qquad D = \{-\ell^2, -(\ell-1)\ell, \dots, -3\ell, -2\ell, -\ell, 0, 1, 2, \dots, \ell\}.$$

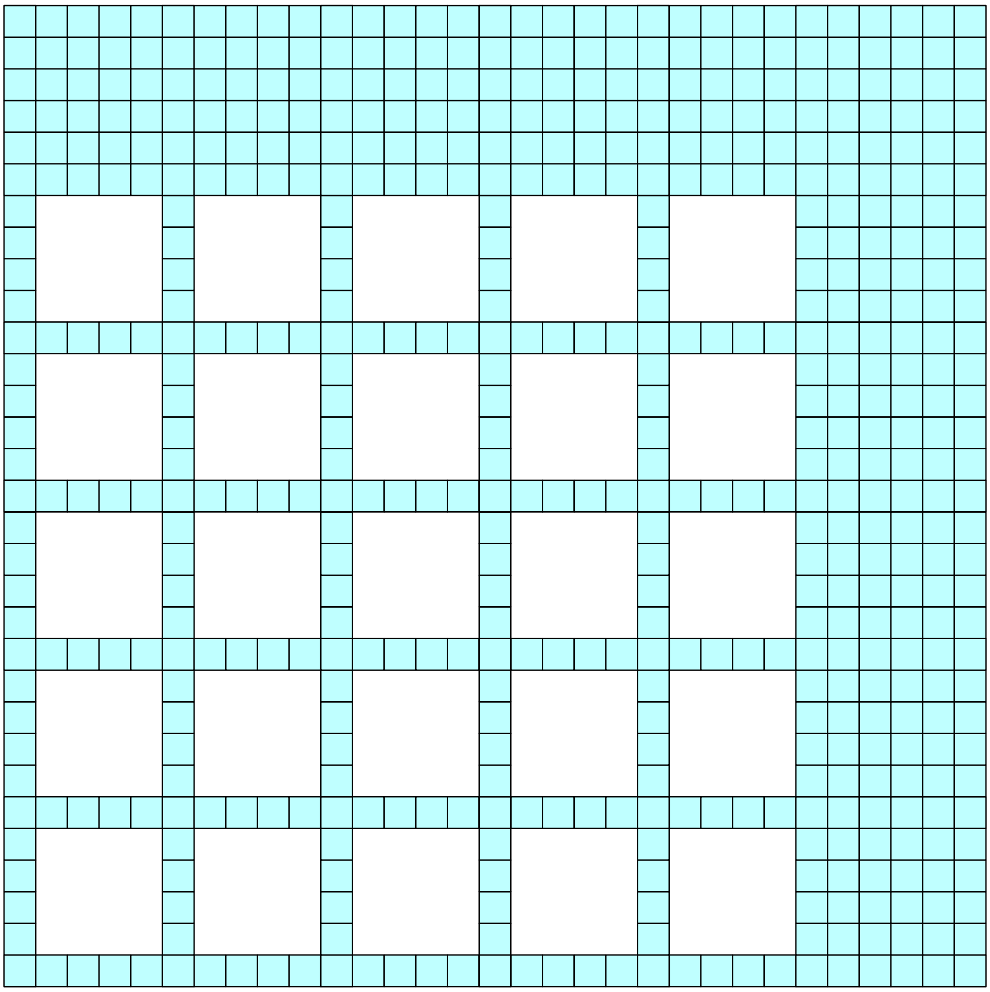
Then $|D|$ is about $\sqrt{2n}$.

We refer to the set D as a *ruler* in the sense that for any $1 \le m < n/2$, there are x_1 and x_2 in D with $x_2 - x_1 = m$. Indeed, one lets x_2 be the remainder when m is divided by ℓ, so that $x_1 = x_2 - m \le 0$ is a multiple of ℓ.

Now, if we let $T = \{-\ell^2, -\ell^2 + 1, \dots, \ell\}$ then we may define

$$S = (D \times T) \cup (T \times D).$$

An illustration below is given for $\ell = 5$.



Note that $|S|$ is at most about $n|D|$, that is, at most about $\sqrt{2} \cdot n^{3/2}$.

Claim. The set S is a bakery of order n.

Proof. Let a and b be any positive integers with $2a + 2b + 4 = n$. By the choice of D, there are x_1 and x_2 in D such that $x_2 - x_1 = a + 1$, as well as y_1 and y_2 in D such that $y_2 - y_1 = b + 1$. Then the bagel with opposite corner cells (x_1, y_1) and (x_2, y_2) has a hole with side lengths a and b and all of its cells are in S, as needed. □

Remark. Let us call a ruler *sparse* when a lot of its marks are missing but we can still measure out each one of the distances 1, 2, ..., N. Then for the set D in the solution essentially we need a sparse ruler with about $c\sqrt{N}$ marks, for some reasonably small positive real constant c. The construction above is simple but also far from optimal. Other constructions are known which are more complicated but yield smaller values of c. See, for example, Ed Pegg Jr, Hitting All The Marks.