USEMO 2021/5

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Twitch Solves ISL

Episode 90

Problem

Given a polynomial $p(x)$ with real coefficients, we denote by $S(p)$ the sum of the squares of its coefficients. For example, $S(20x+21) = 20^2 + 21^2 = 841$.

Prove that if $f(x)$, $g(x)$, and $h(x)$ are polynomials with real coefficients satisfying the identity $f(x) \cdot g(x) = h(x)^2$, then

$$S(f) \cdot S(g) \geq S(h)^2.$$

Video

https://youtu.be/V-9UBJr7aDI

Solution

The following write-up is due to Ankan Bhattacharya, and is the same as the solution proposed by the author.

Claim. Let p be a polynomial with real coefficients, and $n > \deg p$ an integer. Then

$$S(p) = \frac{1}{n} \sum_{k=0}^{n-1} |p(e^{2\pi i k/n})|^2.$$

Proof. Note that

$$|p(e^{2\pi i k/n})|^2 = p(e^{2\pi i k/n}) \cdot p(e^{-2\pi i k/n})$$

so if we define $q(x) = p(x)p(1/x)$, the right-hand side is the sum of q across the nth roots of unity.

Applying a roots of unity filter, the right-hand side is the constant coefficient of $q(x)$. But that constant coefficient is exactly equal to $S(p)$. □

To solve the problem, choose $n > \max\{\deg f, \deg g, \deg h\}$, set $\omega = e^{2\pi i/n}$, and apply the key claim to all three to get that the desired inequality is equivalent to

$$\begin{aligned} &\left[\frac{1}{n} \sum |f(\omega^k)|^2\right] \cdot \left[\frac{1}{n} \sum |g(\omega^k)|^2\right] \ge \left[\frac{1}{n} \sum |h(\omega^k)|^2\right]^2 \\ \iff &\left[\sum |f(\omega^k)|^2\right] \cdot \left[\sum |g(\omega^k)|^2\right] \ge \left[\sum |f(\omega^k)| \cdot |g(\omega^k)|\right]^2. \end{aligned}$$

This is just Cauchy-Schwarz, so we are done.

Remark (Continuous version of above solution)**.** To avoid the arbitrary choice of parameter n, one can make the same argument to show that for any $p \in \mathbb{R}[x]$,

$$S(p) = \frac{1}{2\pi} \int_0^{2\pi} |p(e^{ix})|^2 \, dx.$$

Using Cauchy's inequality for integrals, we obtain a continuous version of the above solution. However, this is technically out of scope for high-school olympiad, despite the fact it is really just the limit as $n \to \infty$ of the above solution.