USEMO 2021/4

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Twitch Solves ISL

Episode 90

Problem

Let ABC be a triangle with circumcircle ω, and let X be the reflection of A in B. Line CX meets ω again at D. Lines BD and AC meet at E, and lines AD and BC meet at F. Let M and N denote the midpoints of AB and AC.

Can line EF share a point with the circumcircle of triangle AMN?

Video

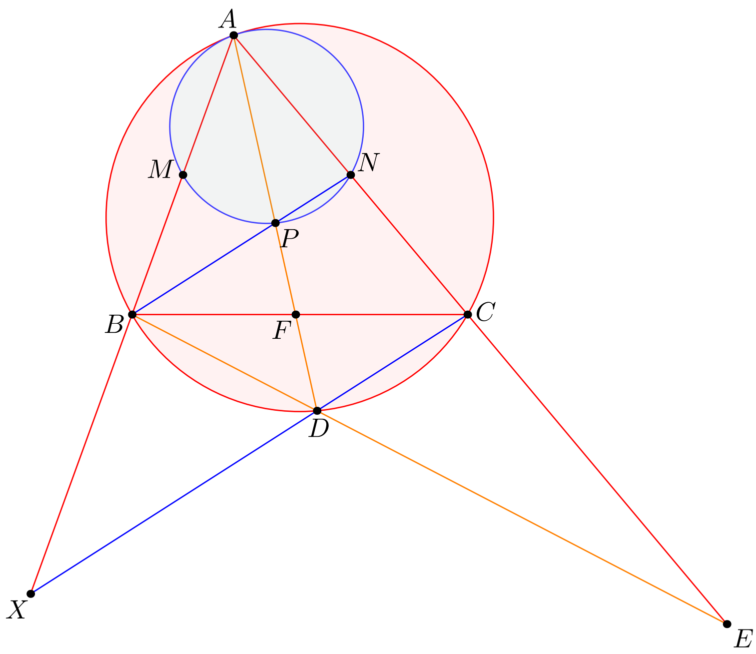
https://youtu.be/V-9UBJr7aDI

Solution

The answer is no, they never intersect.

Classical solution, by author Let P denote the midpoint of $\overline{AD}$, which

- lies on $\overline{BN}$, since $\overline{BN} \parallel \overline{CX}$; and
- lies on (AMN), since it's homothetic to (ABC) through A with factor $\frac{1}{2}$.



Now, note that

$$\measuredangle FBP = \measuredangle CBN = \measuredangle BCD = \measuredangle BAD = \measuredangle BAF \implies FB^2 = FP \cdot FA$$
$$\measuredangle EBN = \measuredangle EDC = \measuredangle BDC = \measuredangle BAC = \measuredangle BAE \implies EB^2 = EN \cdot EA.$$

This means that line EF is the radical axis of the circle centered at B with radius zero, and the circumcircle of triangle AMN. Since B obviously lies outside (AMN), the disjointness conclusion follows.

Projective solution, by Ankit Bisain In this approach we are still going to prove that $\overline{EF}$ is the radical axis of (AMN) and the circle of radius zero at B, but we are not going to use the point P, or even points E and F.

Instead, let $Y = \overline{EF} \cap \overline{AB}$, which by Brokard's theorem on $ABDC$ satisfies $(AB;XY) = -1$. Since $XB = XA$, it follows that $AY : YB = 2$. From here it is straightforward to verify that

$$YB^2 = \frac{1}{9}AB^2 = YM \cdot YA.$$

Thus Y lies on the radical axis.

Finally, by Brokard's theorem again, if O is the center of ω then $\overline{OX} \perp \overline{EF}$. Taking a homothety with scale factor 2 at A, it follows that the line through B and the center of (AMN) is perpendicular to $\overline{EF}$.

Since $\overline{EF}$ contains Y, it now follows that $\overline{EF}$ is the radical axis, as claimed.

Solution with inversion, projective, and Cartesian coordinates, by Ankan Bhattacharya
In what follows, let O be the center of ω. Note that Brokard's theorem gives that $\overline{EF}$ is the polar of X.

Note that since none of E, F, X are points at infinity, O is different from all three.

We consider inversion in ω to eliminate the polar:

- The circumcircle of $\triangle AMN$, i.e. the circle with diameter $\overline{AO}$, is sent to the line ℓ tangent to ω at A.

- The line EF, as the polar of X, is sent to the circle with diameter $\overline{OX}$. (It is indeed a circle, because O does not lie on line EF.)

Thus, if the posed question is true, then we see that ℓ intersects (OX). We claim this is impossible.

Establish Cartesian coordinates with $A = (0, 0)$ and $O = (2, 0)$, so ℓ is the y-axis. Let T be the center of (OX): the midpoint of $\overline{OX}$. Observe:

- B lies on the circle with center $(2, 0)$ and radius 2.

- X lies on the circle with center $(4, 0)$ and radius 4.

- T lies on the circle with center $(3, 0)$ and radius 2.

Thus, let the coordinates of T be (x, y), with $(x - 3)^2 + y^2 = 4$. The intersection of ℓ and (OX) being nonempty is equivalent to

$$\begin{aligned} & d(T,\ell)^2 \le OT^2 \\ \iff\ & x^2 \le (x-2)^2 + y^2 \\ \iff\ & x^2 \le (x-2)^2 + [4-(x-3)^2] \\ \iff\ & (x-1)^2 \le 0, \end{aligned}$$

or $x = 1$ (which forces $y = 0$); i.e. $T = (1, 0)$. However, this forces

$$B = (0, 0) = A,$$

which is not permitted. Thus, line ℓ cannot share a point with (OX), and so line EF cannot share a point with (AMN).