USEMO 2021/3

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Twitch Solves ISL

Episode 89

Problem

Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let O_1 and H_1 denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let O_2 and H_2 denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines O_1O_2 and H_1H_2 are either parallel or coincide.

Video

https://youtu.be/V-9UBJr7aDI

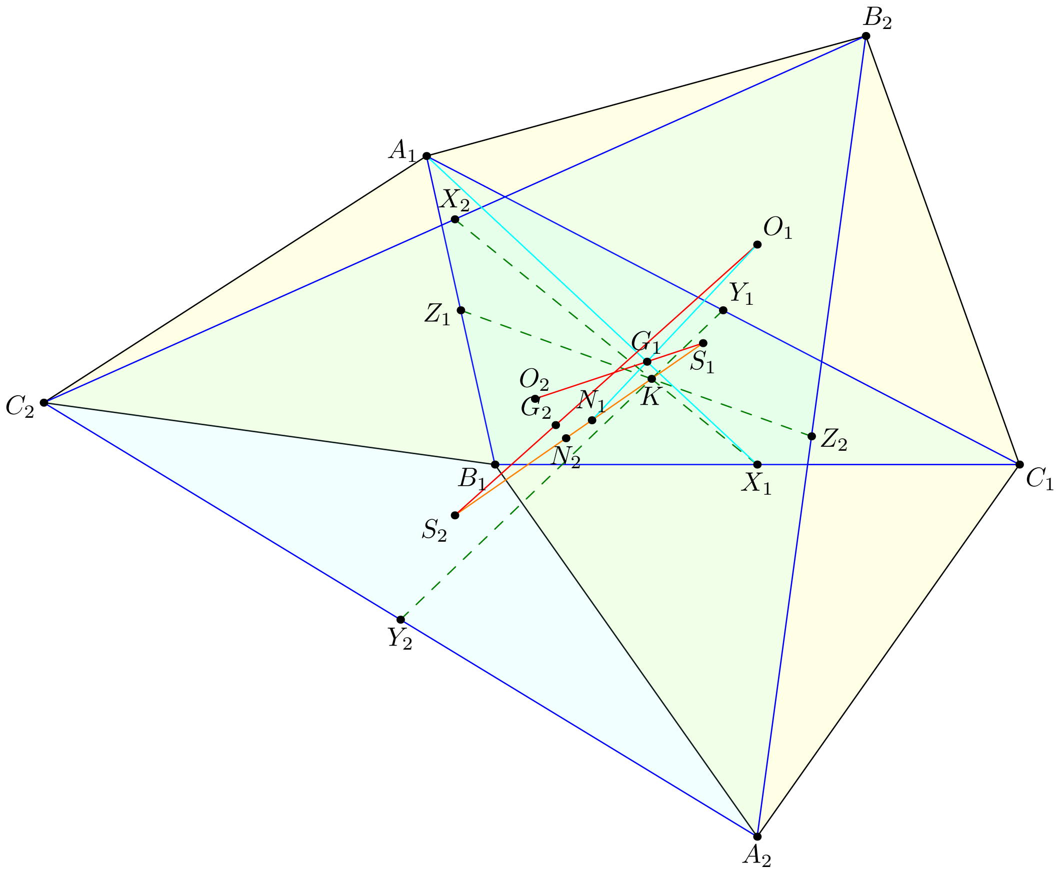
Solution

Let $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ be the medial triangles of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$. The first simple observation is as follows.

Claim. Y_1, Z_1, Y_2, Z_2 are concyclic.

Proof. The distance from each of Y_1, Z_1, Y_2, Z_2 to the midpoint of $\overline{A_1A_2}$ is half the side length of the hexagon. □

Hence by radical axis argument, we obtain that $\overline{X_1X_2}$, $\overline{Y_1Y_2}$, $\overline{Z_1Z_2}$ are concurrent, except possibly when all six points lie on a circle. In this case, $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ share the same nine-point center, so clearly $\overline{O_1O_2} \parallel \overline{H_1H_2}$. So we will assume going forward that $(X_1Y_1Z_1)$ and $(X_2Y_2Z_2)$ are distinct circles.



The heart of the proof revolves around the following two claims.

Claim (Perspectivity)**.** The two triangles $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are perspective at some point K.

Proof. As mentioned above, $\overline{X_1X_2}$, $\overline{Y_1Y_2}$, $\overline{Z_1Z_2}$ are concurrent. □

Let N_1 and G_1 be the circumcenter and centroid of $\triangle X_1Y_1Z_1$; define N_2 and G_2 similarly.

Claim (Orthology)**.** Triangles $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are orthologic. In fact, the orthology center S_1 is the image of O_2 under a homothety centered at G_1 with ratio $-\frac{1}{2}$.

Proof. Since the mentioned homothety takes $\overline{A_1O_2} \to \overline{X_1S_1}$, so

$$\overline{Y_2Z_2} \parallel \overline{B_2C_2} \perp \overline{A_1O_2} \parallel \overline{X_1S_1}$$

as desired. $\square$

We have obtained that $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are both orthologic (with centers S_1 and S_2) and perspective (through K). Hence it follows by **Sondat's theorem** that S_1, S_2, and K lie on a line perpendicular to the perspectrix.

To finish, we follow up with two more claims:

Claim (Perspectrix is radical axis)**.** The perspectrix of the two triangles is exactly the radical axis of their circumcircles, hence perpendicular to $\overline{N_1N_2}$.

Proof. This follows from the earlier observation that Y_1, Y_2, Z_1, Z_2 was cyclic, etc. $\square$

Claim (Degenerate parallelogram)**.** $N_1S_1N_2S_2$ is a (possibly degenerate) parallelogram.

Proof. Because $\overrightarrow{S_1N_2} \overset{O_2}{=} \frac{3}{2}\overrightarrow{G_1G_2} \overset{O_1}{=} \overrightarrow{N_1S_2}$. $\square$

In this way we can conclude that $\overline{N_1N_2} \parallel \overline{S_1S_2}$ through the former claim, but they have the same midpoint by the latter claim, so ultimately all N_i and S_i are collinear.

Finally, note that

$$\overline{N_1N_2} \parallel \overline{N_1S_1} \overset{G_1}{\parallel} \overline{O_1O_2}.$$

It easily follows that $\overline{O_1O_2} \parallel \overline{H_1H_2}$, as wanted.

Remark. An amusing corollary of the above solution is the following:

> Assuming $A_1C_2B_1A_2C_1B_2$ is not self-intersecting, the midpoints of $\overline{A_1A_2}$, $\overline{B_1B_2}$, $\overline{C_1C_2}$ cannot be collinear (unless two of them coincide).

To see this, let M_A, M_B, M_C be said midpoints. If they are different and lie on line ℓ, then $M_BX_1M_CX_2$ is a rhombus with side length $\frac{1}{2}s$, so X_1 and X_2 are reflections in ℓ.

Similarly, $\triangle X_1Y_1Z_1$ and $\triangle X_2Y_2Z_2$ are reflections in ℓ, so $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are as well. This is not possible if $A_1C_2B_1A_2C_1B_2$ is not self-intersecting, because some side will intersect ℓ: then its opposite side will intersect this side at the intersection point.