USEMO 2021/1 Evan Chen

Twitch Solves ISL

Episode 89

Problem

Let n be a positive integer and consider an $n \times n$ grid of real numbers. Determine the greatest possible number of cells c in the grid such that the entry in c is both strictly greater than the average of c's column and strictly less than the average of c's row.

Video

https://youtu.be/kjcY8qQAi5U

External Link

https://aops.com/community/p23517195

Solution

The answer is $(n-1)^2$. An example is given by the following construction, shown for n = 5, which generalizes readily. Here, the lower-left $(n-1) \times (n-1)$ square gives a bound.

$$\begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We give two proofs of the bound. Call a cell *good* if it satisfies the condition of the problem.

Coloring proof, from the author. We now prove that no more than $(n-1)^2$ squares can be good. The cells are weakly ordered by \geq (there may be some ties due to equal elements); we arbitrarily extend it to a total ordering >, breaking all ties. (Alternatively, one can phrase this as perturbing the grid entries in such a way that they become distinct.)

- For every column, we color red the >-smallest element in that column.
- For every row, we color blue the >-largest element in that row.

This means that there are exactly n red and n blue cells. Note that these cells are never good.

Claim. There is at most one cell that is both red and blue.

Proof. Assume for contradiction that P_1 and P_2 are two "purple" cells (both red and blue). Look at the resulting picture

$$\begin{bmatrix} P_1 & x \\ y & P_2 \end{bmatrix}.$$

By construction, we have $P_1 \ge x \ge P_2 \ge y \ge P_1$. This is a contradiction.

Thus at least 2n - 1 cells cannot be good. This proves the bound.

Proof using König's theorem, from Ankan Bhattacharya. This proof is based on the following additional claim:

Claim. No column/row can be all-good, and no transversal can be all-good.

Proof. The first part is obvious. As for the second, let r_i and c_j denote the column sums. If cell (i, j) is good, then

$$r_i < a_{i,j} < c_j.$$

If we have a good transversal, summing the inequality $r_i < c_j$ over the cells in this transversal gives a contradiction (as $\sum r_{\bullet} = \sum c_{\bullet}$).

This claim alone is enough to imply the desired bound.

Claim. There exists a choice of a columns and b rows, with a + b = n + 1, such that no good cells lie on the intersection of the columns and rows.

Proof. Follows by *König's theorem*, and the previous claim. Alternatively, quote the contrapositive of Hall's marriage theorem: because there was no all-good transversal, there must be a set of a columns with more than n - a "compatible" rows.

Suppose $a \leq \frac{n}{2} \leq b$; the other case is similar. Now we bound:

- The $a \times b$ cells of the claim are given to be all non-good.
- In the n a = b 1 remaining rows, there is at least one more non-good cell.

Thus the number of non-good cells is at least

$$ab + (b-1) = (a+1)b - 1 \ge 2 \cdot n - 1 = n^2 - (n-1)^2,$$

and so there are at most $(n-1)^2$ good cells.