USEMO 2021/1

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Twitch Solves ISL

Episode 89

Problem

Let n be a positive integer and consider an $n \times n$ grid of real numbers. Determine the greatest possible number of cells c in the grid such that the entry in c is both strictly *greater* than the average of c's column and strictly *less* than the average of c's row.

Video

https://youtu.be/V-9UBJr7aDI

Solution

The answer is $(n-1)^2$. An example is given by the following construction, shown for $n = 5$, which generalizes readily. Here, the lower-left $(n-1) \times (n-1)$ square gives a bound.

$$\begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We give two proofs of the bound. Call a cell *good* if it satisfies the condition of the problem.

Coloring proof, from the author We now prove that no more than $(n-1)^2$ squares can be good. The cells are weakly ordered by $\geq$ (there may be some ties due to equal elements); we arbitrarily extend it to a total ordering $\gtrdot$, breaking all ties. (Alternatively, one can phrase this as perturbing the grid entries in such a way that they become distinct.)

- For every column, we color red the $\gtrdot$-largest element in that column.
- For every row, we color blue the $\gtrdot$-smallest element in that row.

This means that there are exactly n red and n blue cells. Note that these cells are never good.

Claim. There is at most one cell that is both red and blue.

Proof. Assume for contradiction that P_1 and P_2 are two "purple" cells (both red and blue). Look at the resulting picture

$$\begin{bmatrix} P_1 & x \\ y & P_2 \end{bmatrix}.$$

By construction, we have $P_1 \gtrdot x \gtrdot P_2 \gtrdot y \gtrdot P_1$. This is a contradiction. □

Thus at least $2n-1$ cells cannot be good. This proves the bound.

Proof using König's theorem, from Ankan Bhattacharya This proof is based on the following additional claim:

Claim. No column/row can be all-good, and no transversal can be all-good.

Proof. The first part is obvious. As for the second, let r_i and c_j denote the column sums. If cell (i,j) is good, then

$$r_i < a_{i,j} < c_j.$$

If we have a good transversal, summing the inequality $r_i < c_j$ over the cells in this transversal gives a contradiction (as $\sum r_\bullet = \sum c_\bullet$). □

This claim alone is enough to imply the desired bound.

Claim. There exists a choice of a columns and b rows, with $a + b = n + 1$, such that no good cells lie on the intersection of the columns and rows.

Proof. Follows by *König's theorem*, and the previous claim. Alternatively, quote the contrapositive of Hall's marriage theorem: because there was no all-good transversal, there must be a set of a columns with more than $n-a$ "compatible" rows. $\square$

Suppose $a \le \frac{n}{2} \le b$; the other case is similar. Now we bound:

- The $a \times b$ cells of the claim are given to be all non-good.
- In the $n-a=b-1$ remaining rows, there is at least one more non-good cell.

Thus the number of non-good cells is at least

$$ab + (b-1) = (a+1)b - 1 \ge 2 \cdot n - 1 = n^2 - (n-1)^2,$$

and so there are at most $(n-1)^2$ good cells.