IMO 1999/5

Evan Chen

Twitch Solves ISL

Episode 88

Problem

Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B Lines MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

Video

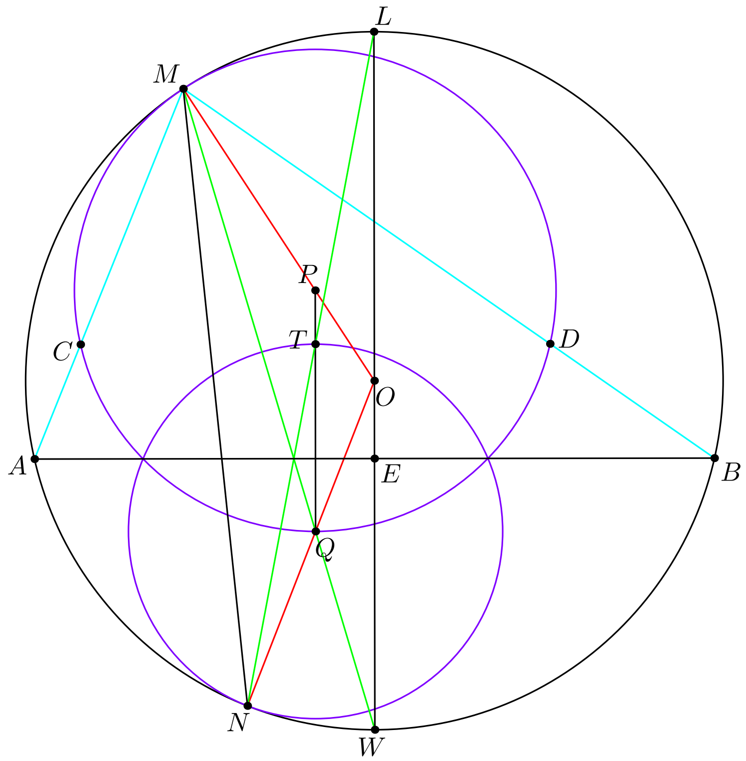
https://youtu.be/V-9UBJr7aDI

Solution

Let P and Q be the centers of Ω_1 and Ω_2.

Let line MQ meet Ω_1 again at W, the homothetic image of Q under $\Omega_1 \to \Omega$.

Meanwhile, let T be the intersection of segment PQ with Ω_2, and let L be its homothetic image on Ω. Since $\overline{PTQ} \perp \overline{AB}$, it follows $\overline{LW}$ is a diameter of Ω. Let O be its center.



Claim. $MNTQ$ is cyclic.

Proof. By Reim: $\measuredangle TQM = \measuredangle LWM = \measuredangle LNM = \measuredangle TNM$. □

Let E be the midpoint of $\overline{AB}$.

Claim. $OEMN$ is cyclic.

Proof. By radical axis, the lines MM, NN, AEB meet at a point R. Then $OEMN$ is on the circle with diameter $\overline{OR}$. □

Claim. MTE are collinear.

Proof. $\measuredangle NMT = \measuredangle TQN = \measuredangle LON = \measuredangle NOE = \measuredangle NME$. □

Now consider the homothety mapping $\triangle WAB$ to $\triangle QCD$. It should map E to a point on line ME which is also on the line through Q perpendicular to $\overline{AB}$; that is, to point T. Hence TCD are collinear, and it's immediate that T is the desired tangency point.