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TWITCH SOLVES ISL

Episode 87

Problem

If a square is partitioned into n convex polygons, determine the maximum number of edges present in the resulting figure.

Video

<https://youtu.be/lt32pF0461U>

Solution

The answer is $E \leq 3n + 1$. One construction is to use parallel stripes. (In fact, we will shortly see any construction which does not use the corners of the square and does not have four edges meeting at a point will achieve this optimal $3n + 1$.)

For the converse, we use the fact that

$$V - E + F = 2$$

in the usual way, with $F = n + 1$ because of the unbounded face. The observation needed is only:

Claim. Every vertex other than possibly the corners of the original square have degree at least 3.

Proof. Follows by convexity for vertices inside the square, and immediate for vertices on the perimeter. \square

The claim then implies

$$2E = \sum_{v \in V} \deg v \geq 2 \cdot 4 + 3(V - 4) = 3V - 4.$$

Combining this with Euler's formula $V - E + F = 2$ gives

$$n - 1 = E - V \geq E - \frac{2E + 4}{3} = \frac{E - 4}{3}$$

which gives the result.