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Twitch Solves ISL

Episode 87

Problem

For a finite set C of integer numbers, we define $S(C)$ as the sum of the elements of C. Find two non-empty sets A and B whose intersection is empty, whose union is the set $\{1, 2, \ldots, 2021\}$ and such that the product $S(A)S(B)$ is a perfect square.

Video

https://youtu.be/4bE3JbYn120

Solution

Use

$$\begin{aligned} B &= \{1, 2, 3, \dots, 243, 1778, 1931, \dots 2020\} \\ A &= \{\text{everything else}\}. \\ S(B) &= 9^2 \cdot 6063 = 243 \cdot 2021 \\ S(A) &= 16^2 \cdot 6063. \end{aligned}$$

Here we were motivated by the deep fact that

$$S(A) + S(B) = 1 + 2 + \cdots + 2021 = 1011 \cdot 2021$$

with $337 = 9^2 + 16^2$ being the only 1 mod 4 prime factor in the right hand side.

Remark. One can show that in fact $S(A)$ and $S(B)$ must be equal to $9^2 \cdot 6063$ and $16^2 \cdot 6063$ via Fermat's Christmas theorem.