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Twitch Solves ISL

Episode 87

Problem

Let a, b, c, x, y, z be real numbers such that

$$\begin{aligned} a^2 + x^2 = b^2 + y^2 = c^2 + z^2 &= (a+b)^2 + (x+y)^2 \\ &= (b+c)^2 + (y+z)^2 = (c+a)^2 + (z+x)^2 \end{aligned}$$

Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

Video

https://youtu.be/KDnDe6KaqL0

Solution

We present two solutions.

Low IQ solution If the common sum is zero, then all numbers are zero. So WLOG the common quantity is 1 by scaling. Write

$$\begin{aligned} 1 &= (a+b)^2 + \left(\pm\sqrt{1-a^2} \pm \sqrt{1-b^2}\right)^2 \\ &= a^2 + 2ab + b^2 + 2 - (a^2+b^2) \pm 2\sqrt{(1-a^2)(1-b^2)} \\ \implies 2ab+1 &= 2\sqrt{(1-a^2)(1-b^2)} \\ \implies 4a^2b^2 + 4ab + 1 &= 4(1-a^2)(1-b^2) = 4 - 4a^2 - 4b^2 + 4(ab)^2 \\ \implies 4ab &= 4(1-a^2)(1-b^2) = 3 - 4a^2 - 4b^2 \\ \implies a^2 + ab + b^2 &= \frac{3}{4}. \end{aligned}$$

Proceeding in the same way we get

$$b^2 + bc + c^2 = \frac{3}{4}, \quad c^2 + ca + a^2 = \frac{3}{4}.$$

By using HMMT 2014 Problem A-9, this implies $ab + bc + ca$ is equal to some absolute constant k (not depending on the choice of solution to the system). Hence, adding gives

$$a^2 + b^2 + c^2 = \frac{9}{4} - \frac{1}{2}k.$$

Similarly, $x^2 + y^2 + z^2 = \frac{9}{4} - \frac{1}{2}k$, as needed.

Remark. This solution only uses the real hypothesis in the first step when scaling the common quantity to 1. Thus, as long as $a^2 + x^2 \neq 0$, the problem still holds even in the complex setting.

On the other hand, the tuple $(1, 1, 1, i, i, i)$ is a counterexample to the statement without the real hypothesis.

High IQ solution As before scale so that the common sum is 1. Consider

$$z_1 = a + xi, \qquad z_2 = b + yi, \qquad z_3 = c + zi$$

in the complex plane. The problem hypothesis says that the vectors are on the unit circle such that $|z_i + z_j| = 1$ for $i \neq j$, which implies they are the vertices of an equilateral triangle. Hence

$$0 = z_1^2 + z_2^2 + z_3^2$$

and the real part of the right-hand side is $(a^2 + b^2 + c^2) - (x^2 + y^2 + z^2)$.