Iberoamerican 2021/3

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Twitch Solves ISL

Episode 87

Problem

Let a_1, a_2, a_3, ... be a sequence of positive integers and let b_1, b_2, b_3, ... be the sequence of real numbers given by

$$b_n = \frac{a_1 a_2 \cdots a_n}{a_1 + a_2 + \cdots + a_n}, \text{ for } n \geq 1.$$

Show that, if there exists at least one term among every million consecutive terms of the sequence $b_1, b_2, b_3, \dots$ that is an integer, then there exists some k such that $b_k > 2021^{2021}$.

Video

https://youtu.be/Wx4-fscAMB8

Solution

Let $K = 10^{10^{100}}$ be a huge absolute constant. We consider two cases. The first case is captured in the following claim.

Claim. Suppose that after the first K terms, at least one of every K terms a_i is greater than 1. Then b_i is unbounded.

Proof. In that case, consider the first n terms for some large n. Let $M = \max(a_1, \dots, a_n)$. We have that

$$b_n \geq \frac{2^{\lfloor 10^{-6}n-1 \rfloor} M}{M + M + \cdots + M} \geq \frac{1}{n} 2^{\lfloor 10^{-6}n-1 \rfloor}$$

since the product of every million terms is at least 2 and one of the products is actually equal to M. The right-hand side is unbounded in n, as desired. $\square$

Thus, we may focus on the case where $a_N = a_{N+1} = \cdots = a_{N+K-1} = 1$ for some $N \geq K$. Look at two indices b_{N+i} and b_{N+j} with $|j - i| \leq 10^6$ and such that b_i and b_j are integers, as promised by the problem condition. We may write

$$b_i = \frac{X}{Y+i}, \qquad b_j = \frac{X}{Y+j}$$

where $Y = a_1 + \cdots + a_N \geq N \geq K$. Then

$$X \geq \operatorname{lcm}(Y+i, Y+j) = \frac{(Y+i)(Y+j)}{\gcd(Y+i, Y+j)} \geq \frac{1}{10^6}(Y+i)(Y+j)$$

which means that

$$b_j \geq \frac{Y+j}{10^6} > \frac{K}{10^6}$$

as needed.