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Twitch Solves ISL

Episode 87

Problem

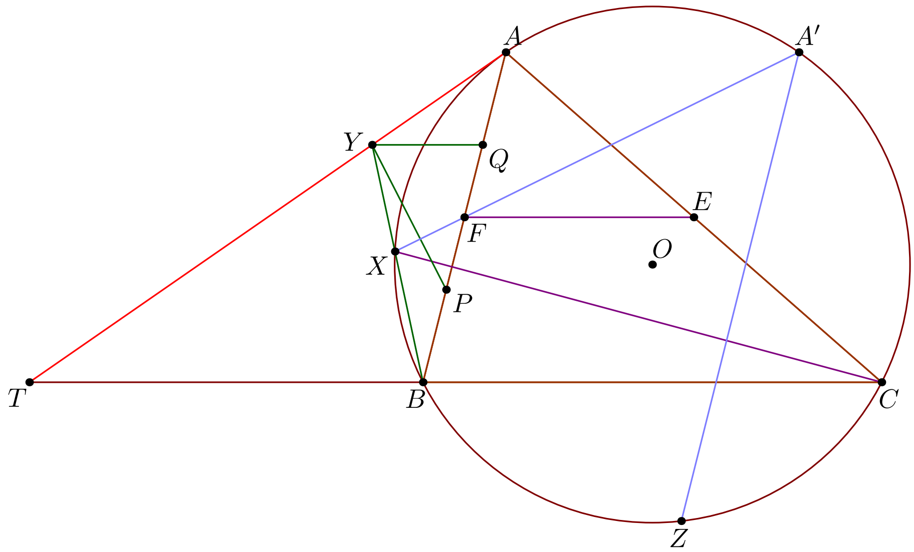
Consider an acute-angled triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of the triangle CEF and Γ meet at X and C, with $X \neq C$. The line BX and the tangent to Γ through A meet at Y. Let P be the point on segment AB so that $YP = YA$, with $P \neq A$, and let Q be the point where AB and the parallel to BC through Y meet each other. Show that F is the midpoint of PQ.

Video

https://youtu.be/vdvTg5tLx4I

Solution

Let $T = \overline{AA} \cap \overline{BC}$.



Claim (Main claim)**.**

$$\frac{AX}{BX} = \frac{AT}{AB}.$$

Proof. Let A' denote the point such that $ABCA'$ is an isosceles trapezoid with $AA' \parallel BC$. Note that A' lies on line FX, because $\measuredangle CXF = \measuredangle CEF = \measuredangle AEF = \measuredangle ACB$.

Now let Z be the harmonic conjugate of X with respect to AB. Projecting $AXBZ$ through A' onto line AB we find $\overline{A'Z} \parallel \overline{AB}$.

Finally from $\triangle BAT \sim \triangle A'AC$, we get

$$\frac{AX}{XB} = \frac{AZ}{ZB} = \frac{BA'}{AA'} = \frac{AC}{AA'} = \frac{AT}{AB}.$$ □

Since

$$\measuredangle APY = \measuredangle YAP = \measuredangle YAB = \measuredangle ACB = \measuredangle AXB = \measuredangle AXY$$

we get $YXPA$ is cyclic and hence

$$BP \cdot BA = BX \cdot BY = BX \cdot \frac{AB \cdot AY}{AX}$$

with the last equality due to $\triangle BYA \sim \triangle AYX$. Meanwhile,

$$AQ = \frac{AY}{AT} \cdot AF = \frac{AY}{AT} \cdot AB.$$

Combining with the main claim, we have $AQ = BP$ as needed.