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Twitch Solves ISL

Episode 87

Problem

Let $P = \{p_1, p_2, \dots, p_{10}\}$ be a set of 10 different prime numbersand let A be the set of all the integers greater than 1 so that their prime decomposition only contains primes of P. The elements of A are colored in such a way that:

- each element of P has a different color,
- if $m, n \in A$, then mn is the same color of m or n,
- for any pair of different colors $\mathcal{R}$ and $\mathcal{S}$, there are no $j, k, m, n \in A$ (not necessarily distinct from one another), with j, k colored $\mathcal{R}$ and m, n colored $\mathcal{S}$, so that j is a divisor of m and n is a divisor of k, simultaneously.

Prove that there exists a prime of P so that all its multiples in A are the same color.

Video

https://youtu.be/Wx4-fscAMB8

Solution

By abuse of notation, we identify colors as subsets of A.

Let's say color $\mathcal{A}$ is *recessive* to color $\mathcal{B}$ if there exists $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a \mid b$. We write this as $\mathcal{A} \lessdot \mathcal{B}$.

Claim (Anti-symmetry)**.** We can't have both $\mathcal{A} \lessdot \mathcal{B}$ and $\mathcal{B} \lessdot \mathcal{A}$.

Proof. This is the third condition. □

Claim. If $\mathcal{A} \lessdot \mathcal{B}$, then for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we have $ab \in \mathcal{B}$.

Proof. By second condition, we know ab is colored either $\mathcal{A}$ or $\mathcal{B}$, but the former case would cause $\mathcal{B} \lessdot \mathcal{A}$ since $b \mid ab$. □

Claim (Transitivity)**.** If $\mathcal{A} \lessdot \mathcal{B}$, and $\mathcal{B} \lessdot \mathcal{C}$, then $\mathcal{A} \lessdot \mathcal{C}$.

Proof. Suppose $a \mid b$ and $b' \mid c$ witnesses the first two relations. Now, $bb' \in \mathcal{B}$, and $bc \in \mathcal{C}$. So $a \mid bb' \mid bc$ and we get $\mathcal{A} \lessdot \mathcal{C}$. □

It follows $\lessdot$ is a total order. So it has a maximal element, that is, a color $\mathcal{M}$ which is not recessive to any other. Let $p \in \mathcal{M}$ be the prime of P in $\mathcal{M}$, promised by the first condition. It now follows all multiples of p are in $\mathcal{M}$, via the second claim.