China 2018/3

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Twitch Solves ISL

Episode 86

Problem

Let q be a positive integer which is not a perfect cube. Prove that there exists a positive constant C such that for all natural numbers n, one has

$$\{nq^{\frac{1}{3}}\} + \{nq^{\frac{2}{3}}\} \geq Cn^{-\frac{1}{2}}$$

where $\{x\}$ denotes the fractional part of x.

Video

https://youtu.be/sCZMl8JGDh0

External Link

https://aops.com/community/p9374386

Solution

Assume for contradiction there is a bad n. Define the integers

$$A = nq^{1/3} - \delta$$
$$B = nq^{2/3} - \varepsilon$$

for small δ and ϵ in $(0, Cn^{-1/2})$. First, note that we have

$$\begin{aligned}
\mathbb{Z} \ni A^3 - n^3 q &= -3n^2 q^{2/3}\delta + 3nq^{1/3}\delta^2 - \delta^3 \\
\mathbb{Z} \ni B^3 - n^3 q^2 &= -3n^2 q^{4/3}\varepsilon + 3nq^{2/3}\varepsilon^2 - \varepsilon^3 \\
\mathbb{Z} \ni AB - n^2 q &= -nq^{2/3}\delta - nq^{1/3}\varepsilon + \delta\varepsilon.
\end{aligned}$$

(For clarity, we highlighted in red terms which are bigger than $O(1)$.) Now we define two integers X and Y such that

$$\begin{aligned}
X &= (A^3 - n^3 q) - 3n(AB - n^2 q) \\
&= 3nq^{1/3}\delta^2 - \delta^3 + 3n^2 q^{1/3}\varepsilon - 3n\delta\varepsilon. \\
Y &= qX + (B^3 - n^3 q) \\
&= 3nq^{4/3}\delta^2 - q\delta^3 - 3nq\delta\varepsilon + 3nq^{2/3}\varepsilon^2 - \varepsilon^3 \\
&= 3n\left(q^{4/3}\delta^2 - q\delta\varepsilon + q^{2/3}\varepsilon^2\right) - \varepsilon^3 - q\delta^3.
\end{aligned}$$

Because δ and ε are $O(n^{-1/2})$, it is evident that $Y < 1$ for sufficiently large n. However, we contend $Y > 0$ for sufficiently large n as well:

$$\begin{aligned}
Y &= 3n\left(q^{4/3}\delta^2 - q\delta\varepsilon + q^{2/3}\varepsilon^2\right) - \varepsilon^3 - q\delta^3. \\
&\overset{\text{AMGM}}{\geq} \frac{3}{2}n(q^{4/3}\delta^2 + q^{2/3}\varepsilon^2) - \varepsilon^3 - q\delta^3 \\
&= \delta^2\left(\frac{3}{2}nq^{4/3} - q\delta\right) + \varepsilon^2\left(\frac{3}{2}nq^{2/3} - \varepsilon\right)
\end{aligned}$$

which is positive for large n. Since Y was supposed to be an integer, $0 < Y < 1$ is a contradiction.