# China 2018/3

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Twitch Solves ISL

Episode 86

## Problem

Let $q$ be a positive integer which is not a perfect cube. Prove that there exists a positive constant $C$ such that for all natural numbers $n$, one has

$$\{nq^{\frac{1}{3}}\} + \{nq^{\frac{2}{3}}\} \geq Cn^{-\frac{1}{2}}$$

where $\{x\}$ denotes the fractional part of $x$.

## Video

https://youtu.be/sCZMl8JGDh0

## Solution

We'll take $C = \frac{1}{1000q}$ and show it works for large $n$.

Assume there is a bad $n$. Define the integers

$$A = nq^{1/3} - \delta$$
$$B = nq^{2/3} - \varepsilon$$

for small $\delta$ and $\epsilon$ in $(0, Cn^{-1/2})$. Compute

$$\mathbb{Z} \ni A^3 - n^3q = -3\delta n^2 q^{1/3} + 3n^2q^{2/3}\delta - \delta^3$$
$$\mathbb{Z} \ni A^2 = n^2q^{2/3} - 2\delta nq^{1/3} + \delta^2$$
$$\mathbb{Z} \ni B^3 - n^3q^2 = -3\varepsilon n^2 q^{2/3} + 3n^2q^{4/3}\varepsilon - \varepsilon^3$$
$$\mathbb{Z} \ni B^2 = n^2q^{4/3} - 2\varepsilon nq^{2/3} + \varepsilon^2$$
$$\mathbb{Z} \ni AB = -\delta nq^{2/3} - n\varepsilon q^{1/3} + \delta\varepsilon.$$

Define more integers

$$X = 2(A^3 - n^3q) - 3nA^2 = 6n^2q^{2/3}\delta - 2\delta^3 - 3n^3q^{2/3} - 3n\delta^2$$
$$Y = 2(B^3 - n^3q^2) - 3nB^2 = 6n^2q^{4/3}\varepsilon - 2\varepsilon^3 - 3n^3q^{4/3} - 3n\varepsilon^2$$
$$Z = qX + Y + 6nAB = -2\delta^3 q - 3n^3q^{5/3} - 3nq\delta^2 - 2\varepsilon^3 - 3n^3q^{4/3} - 3n\varepsilon^2$$
$$W = Z + 3n^2qB + 3n^2qA = -2\delta^3 q - 3n^2q\varepsilon - 3nq\delta^2 - 2\varepsilon^3 - 3n^2q\delta - 3n\varepsilon^2$$

However the condition on $\delta$ and $\epsilon$ causes $W < 0$ and $W > -1$, contradiction.