

MR U552

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TWITCH SOLVES ISL

Episode 85

Problem

Find all polynomials with real coefficients P which satisfy

$$\begin{aligned} P(P(a+b)) - 2ab(2P(a+b) - ab) &\geq P(a^2) + P(b^2) \\ &\geq P(a^2 + b^2) - P(ab\sqrt{2}) \end{aligned}$$

for all $a, b \in \mathbb{R}$.

Video

<https://youtu.be/i1atXI9YdxE>

Solution

The answers are:

- $P(x) = cx^2$ for $-1 \leq c \leq 0$;
- $P(x) = x^2$;
- $P(x) = x^2 + \ell$ for $\ell \geq 1$.

Claim. We have $\deg P \leq 2$.

Proof. Assume that $\deg P > 2$. Comparing the leading terms when $a = b$, we find that the leading coefficient of the dominating $P(P(a+b))$ is positive. However, plugging in $a = -b$ now gives a contradiction. \square

Claim. The x^1 coefficient of P is zero and $P(0) \geq 0$.

Proof. Let $P(x) = cx^2 + kx + \ell$. Expand the eastern inequality:

$$ca^4 + ka^2 + \ell + cb^4 + kb^2 + \ell \geq c(a^4 + b^4) + k(a^2 + b^2) - kab\sqrt{2}.$$

Mass cancellation boils this down to

$$2\ell \geq -kab\sqrt{2}.$$

Since this is valid for all $a, b \in \mathbb{R}$, this means $k = 0$ and $\ell \geq 0$. \square

Finally, put $P(x) = cx^2 + \ell$ and the western inequality reads

$$c^3(a+b)^4 + 2\ell c^2(a+b)^2 + \ell^2 c + 2a^2 b^2 \geq c(a^4 + b^4) + \ell + 4c \cdot ab(a+b)^2 + 4ab \cdot \ell.$$

We make some deductions:

- Let $b = 0$ and $a \rightarrow \infty$ to get $c^3 \geq c$.
- Let $a = -b = x$ gives

$$P(P(0)) + 4x^2 P(0) + 2x^4 \geq 2P(x^2)$$

which means $c \leq 1$.

- Combining these two, we get $c \in [-1, 0] \cup \{1\}$.
- Let $a = b = 0$ to get $\ell^2 c \geq \ell$. Since $\ell \geq 0$ is known, either $\ell = 0$ or $\ell c \geq 1$. The latter requires $c > 0$, or $c = 1$.

Hence, we obtain that either

$$(c \in [-1, 0] \cup \{1\} \text{ and } \ell = 0) \text{ or } (c = 1 \text{ and } \ell \geq 1)$$

which gives the answer set. One can manually check this works by plugging it in and bashing some more (details left to excessively bored reader).