MR U552

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Twitch Solves ISL

Episode 85

Problem

Find all polynomials with real coefficients P which satisfy

$$\begin{aligned} P(P(a+b)) - 2ab(2P(a+b) - ab) &\geq P(a^2) + P(b^2) \\ &\geq P(a^2+b^2) - P(ab\sqrt{2}) \end{aligned}$$

for all $a, b \in \mathbb{R}$.

Video

https://youtu.be/i1atXI9YdxE

Solution

Claim. We have $\deg P \le 2$.

Proof. Assume that $\deg P > 2$. Comparing the leading terms when $a = b$, we find that the leading coefficient of the dominating $P(P(a+b))$ is positive. However, plugging in $a = -b$ now gives a contradiction. □

Claim. The x^1 coefficient of P is zero and $P(0) \ge 0$.

Proof. Let $P(x) = cx^2 + kx + \ell$. Expand the eastern inequality:

$$ca^4 + ka^2 + \ell + cb^4 + kb^2 + \ell \ge c(a^4 + b^4) + k(a^2 + b^2) - kab\sqrt{2}.$$

Mass cancellation boils this down to

$$2\ell \ge -kab\sqrt{2}.$$

Since this is valid for all $a, b \in \mathbb{R}$, this means $k = 0$ and $\ell \ge 0$. □

Finally, put $P(x) = cx^2 + \ell$ and the western inequality reads

$$c^3(a+b)^4 + 2\ell c^2(a+b)^2 + \ell^2 c + 2a^2b^2 \ge c(a^4 + b^4) + \ell + 4c \cdot ab(a+b)^2 + 4ab \cdot \ell.$$

We make some deductions:

- Let $b = 0$ and $a \to \infty$ to get $c^3 \ge c$.
- Let $a = -b = x$ gives
  $$P(P(0)) + 4x^2 P(0) + 2x^4 \ge 2P(x^2)$$
  which means $c \le 1$.
- Combining these two, we get $c \in [-1, 0] \cup \{1\}$.
- Let $a = b = 0$ to get $\ell^2 c \ge \ell$. Since $\ell \ge 0$ is known, either $\ell = 0$ or $\ell c \ge 1$. The latter requires $c > 0$, or $c = 1$.

Hence, we obtain that either

$$(c \in [-1, 0] \text{ and } \ell = 0) \text{ or } (c = 1 \text{ and } \ell \ge 1)$$

which gives the answer set. One can manually check this works by plugging it in and bashing some more (details left to excessively bored reader).