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Twitch Solves ISL

Episode 84

Problem

Let ABC be a triangle, and M an interior point such that $\angle MAB = 10^\circ$, $\angle MBA = 20^\circ$, $\angle MAC = 40^\circ$ and $\angle MCA = 30^\circ$. Prove that the triangle is isosceles.

Video

https://youtu.be/HYHsVSqveMI

Solution

Let $\theta = \angle MBC < 80^\circ$. By trig Ceva, we get

$$\frac{\sin 10^\circ}{\sin 40^\circ} \cdot \frac{\sin \theta}{\sin 20^\circ} \cdot \frac{\sin 30^\circ}{\sin(80^\circ - \theta)} = 1.$$

This simplifies to

$$\sin \theta = 4 \sin(80^\circ - \theta) \sin 40^\circ \cos 10^\circ.$$

Claim. We have $\theta = 60^\circ$.

Proof. The left-hand side is increasing in θ and the right-hand side is decreasing in θ, so at most one value of θ works. But we also have

$$\begin{aligned} 4 \sin 20^\circ \sin 40^\circ \cos 10^\circ &= 2 \left(\cos 20^\circ - \cos 60^\circ\right) \cos 10^\circ \\ &= 2 \cos 20^\circ \cos 10^\circ + \sin 80^\circ \\ &= \left(\cos 30^\circ - \cos 10^\circ\right) + \sin 80^\circ = \cos 30^\circ \end{aligned}$$

as desired. □