# Shortlist 2020 N5 <br> Evan Chen 

## Twitch Solves ISL

Episode 83

## Problem

Determine all functions $f: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{\geq 0}$ satisfying the two conditions:

- $f(x y)=f(x)+f(y)$ for every positive integer $x$ and $y$;
- there are infinitely many positive integers $n$ such that $f(k)=f(n-k)$ for all $k<n$.


## Video

https://youtu.be/3B15SRkbyJg

## External Link

https://aops.com/community/p22698553

## Solution

The following solution was given by Ankan Bhattacharya.
The answer is and $f(x) \equiv c \nu_{p}(x)$ for any prime $p$ and constant $c \geq 0$. These obviously work.

For the other direction, let $\mathcal{M}$ be the set of integers satisfying the second condition.
Claim. $\mathcal{M}$ is closed under division.
Proof. Tautological.
Now assuming $f$ is not identically zero, fix $p$ the smallest prime for which $f(p)>0$.
Claim. This prime $p$ divides any element of $\mathcal{M}$ which is greater than $p$.
Proof. Suppose $m \in \mathcal{M}$ and $m>p$. Use division algorithm to get $m=p \cdot s+r$ where $0 \leq r<p$. If $r>0$, then

$$
f(r)=f(p \cdot s)=f(p)+f(s)>0
$$

a contradiction to minimality of $p$.
Claim. Every element of $\mathcal{M}$ equals a power of $p$ times an integer less than or equal to $p$.
Proof. Follows from the last two claims.
Assume for contradiction $q$ is a prime greater than $p$ and with $f(q)>0$. For some $p^{e} \in \mathcal{M}$ exceeding $q$, use the division algorithm to get

$$
p^{e}-q s=r<q
$$

and hence

$$
f(q)+f(s)=f(r), \quad r<q .
$$

But evidently $\nu_{p}(s)=\nu_{p}(r)$ and $\nu_{q}(r)=0$, hence $f(s)=f(r)$, contradiction.

