Shortlist 2020 N5

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Twitch Solves ISL

Episode 83

Problem

Determine all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\ge 0}$ satisfying the three conditions:

- $f(xy) = f(x) + f(y)$ for every positive integer x and y;
- there are infinitely many positive integers n such that $f(k) = f(n-k)$ for all $k < n$.

Video

https://youtu.be/3Bl5SRkbyJg

Solution

The following solution was given by Ankan Bhattacharya.

The answer is and $f(x) \equiv c\nu_p(x)$ for any prime p and constant $c \ge 0$. These obviously work.

For the other direction, let $\mathcal{M}$ be the set of integers satisfying the sum condition.

Claim. $\mathcal{M}$ is closed under division.

Proof. Tautological. □

Now assuming f is not identically zero, fix p the smallest prime for which $f(p) > 0$.

Claim. This prime p divides any element of $\mathcal{M}$ which is greater than p.

Proof. Suppose $m \in \mathcal{M}$ and $m > p$. Use division algorithm to get $m = p \cdot s + r$ where $0 \le r < p$. If $r > 0$, then

$$f(r) = f(p \cdot s) = f(p) + f(s) > 0$$

a contradiction to minimality of p. □

Claim. Every element of $\mathcal{M}$ equals a power of p times an integer less than or equal to p.

Proof. Follows from the last two claims. □

Assume for contradiction q is a prime greater than p and with $f(q) > 0$. For some $p^e \in M$ exceeding q, use the division algorithm to get

$$p^e - qs = r < q$$

and hence

$$f(q) + f(s) = f(r), \qquad r < q.$$

But evidently $\nu_p(s) = \nu_p(r)$ and $\nu_q(r) = 0$, hence $f(s) = f(r)$, contradiction.