USAMO 1996/3

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Twitch Solves ISL

Episode 81

Problem

Let ABC be a triangle. Prove that there is a line ℓ (in the plane of triangle ABC) such that the intersection of the interior of triangle ABC and the interior of its reflection $A'B'C'$ in ℓ has area more than $\frac{2}{3}$ the area of triangle ABC.

Video

https://youtu.be/30XpnQ__wzs

Solution

All that's needed is:

Claim. If ABC is a triangle where $\frac{1}{2} < \frac{AB}{AC} < 1$, then the $\angle A$ bisector works.

Proof. Let the $\angle A$-bisector meet BC at D. The overlapped area is $2[ABD]$ and

$$\frac{[ABD]}{[ABC]} = \frac{BD}{BC} = \frac{AB}{AB + AC}$$

by angle bisector theorem. □

In general, suppose $x < y < z$ are sides of a triangle. Then $\frac{1}{2} < \frac{y}{z} < 1$ by triangle inequality as needed.